Value at Risk / Expected Shortfall

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Risk Measure

In the following, the random variable $X$ stands for the potential loss due to various risks.

Risk Measure

A risk measure is a mapping that assigns a value $V_X$ to $X$.

Coherent Risk Measure

A risk measure $V$ is said to be coherent if it satisfies the following properties for r.v.s. $X$, $Y$:

- (Subadditivity) $V_{X+Y} \leq V_X + V_Y$
- (Monotonicity) $X \leq Y \implies V_X \leq V_Y$
- (Homogeneity) $V_{\lambda X} = \lambda V_X, \lambda > 0$
- (Translation Invariance) $V_{X+\mu} = V_X + \mu, \mu > 0$
Subadditivity means that, the combined risk of several portfolios is lower than the sum of risks of those portfolios, as should happen with portfolio diversification.

Example

The expectation $EX$ of a random variable $X$ is a coherent risk measure.
Cumulative Distribution Function

The cumulative distribution function (cdf) $F_X$ of a r.v. $X$ is

$$F_X(x) \equiv P(X \leq x)$$

Empirical Cumulative Distribution Function

$$F_N(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{x_i \leq x\}}, \quad x \in \mathbb{R}$$
Value at Risk

The Value at Risk has two objectives:

- to provide a risk measure
- to determine an adequate level of capital reserves that matches the current level of risk

In other words, risk managing means determining a level $V_X$ (of capital requirement) that will not be “too much” exceed by $X$. For example, setting $V_X$ such that

$$P(X \leq V_X) \geq 0.95, \text{ i.e. } P(X > V_X) \leq 0.05$$

means that insolvency will occur with probability less than 5%.
Value at Risk

The Value at Risk (VaR) at the level $p \in (0, 1)$ (or $p$-quantile risk measure) is defined as

$$V_X^p \equiv \inf\{x \in \mathbb{R} : P(X \leq x) \geq p\}$$

- $X$ is lower than $V_X^p$ with probability at least $p$.
- $V_X^p$ can be negative, which indicates a profit.
- (Caveat) $V_X^p$ does not contain any information on how large losses can be beyond $V_X^p$!
Figure: Two distributions having the same Value at Risk $\sqrt{\chi^2_{0.95}} = 2.145$
Figure: DJIA Market returns v.s. normalized Gaussian returns, which tends to underestimate the probabilities of extreme events.
The Tail Value at Risk

The Tail Value at Risk at the level $p$ is defined by

$$\eta_X^p \equiv \frac{1}{1 - p} \int_p^1 V_X^q \, dq$$

The Conditional Tail Expectation

The Conditional Tail Expectation at the level $p$ is defined by

$$\text{CTE}_X^p \equiv \mathbb{E}\{X \mid X > V_X^p\} = \frac{\mathbb{E}\left\{X \mathbb{1}_{\{X > V_X^p\}}\right\}}{P(X > V_X^p)}$$
The Expected Shortfall

The Expected Shortfall at the level $p$ is defined by

$$E^p_X \equiv \frac{1}{1-p} \mathbb{E} \left\{ X \mathbf{1}_{\{X \geq V^p_X\}} \right\} + \frac{V^p_X}{1-p} (1 - p - P(X \geq V^p_X))$$

Propositions

- $V^p_X$ is coherent when $X$ is Gaussian.
- If $P(X = V^p_X) = 0$, then $\text{CTE}^p_X = \eta^p_X$.
- $E^p_X = \eta^p_X$.
- $E^p_X$ and $\eta^p_X$ are coherent risk measures.
Example

Consider r.v. $X \in \{10, 100, 150\}$ with

$$P(X = 10) = 0.96, \ P(X = 100) = 0.03, \ P(X = 150) = 0.01.$$  

Compute $V^{0.98}_X, \eta^{0.98}_X, \text{CTE}^{0.98}_X, \ E^{0.98}_X$.

We have

$$V^{0.98}_X = 100,$$

$$\eta^{0.98}_X = \frac{1}{0.02} ((0.99 - 0.98) \times 100 + (1 - 0.99) \times 150) = 125,$$

$$\text{CTE}^{0.98}_X = \frac{1}{0.01} \times 150 \times 0.01 = 150,$$

$$E^{0.98}_X = \frac{1}{0.02} (100 \times 0.03 + 150 \times 0.01) + \frac{100}{0.02} (0.02 - (0.03 + 0.01))$$

$$= 125.$$
Figure: R Program / RStudio in Action
Figure: TWII stock returns
Figure: Empirical cumulative distribution function of TWII returns
**Figure**: Historical v.s. Gaussian estimates of Value at Risk
Figure: Historical v.s. Gaussian estimates of Expected Shortfall
**Figure**: Historical Value at Risk v.s. Historical Expected Shortfall