The Monotonicity Approach to Inverse Obstacle Scattering Problems

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1 Notations, Definitions and Prerequisites

Definition 1. (c.f. Grisvard [6]) Let Ω be an open subset in $\mathbb{R}^n$. The boundary of Ω, denoted by Γ, is $C^{k,1}$ (resp. Lipchitz) if for $x \in \Gamma$ there exists a neighborhood V of x and new orthogonal coordinates $\{y_1, y_2, \ldots, y_n\}$ such that

1. V is an hypercube in the new coordinates:

$$V = \{(y_1, y_2, \ldots, y_n) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n\}$$

2. There exists a $C^{k,1}$ (resp. Lipschitz) function φ, defined in

$$V' = \{(y_1, y_2, \ldots, y_{n-1}) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n-1\}$$

such that

$$\begin{aligned} &|\varphi(y')| \leqslant \frac{a_n}{2} \quad \forall y' = (y_1, y_2, \ldots, y_{n-1}) \in V' \\ &\Omega \cap V = \{y = (y', y_n) \in V |\, y_n < \varphi(y')\} \\ &\Gamma \cap V = \{y = (y', y_n) \in V |\, y_n = \varphi(y')\} \end{aligned}$$

Proposition 1 (Vector Green Formula)**.**

$$\begin{aligned} &\int_\Omega (E \cdot \Delta H - H \cdot \Delta E)\, \mathrm{d}V \\ &\qquad = \int_\Gamma (E \times \operatorname{curl} H + E \operatorname{div} H - H \times \operatorname{curl} E - H \operatorname{div} E) \cdot \nu \,\mathrm{d}\sigma \end{aligned}$$

If $\operatorname{div} E = \operatorname{div} H = 0$, then

$$\begin{aligned} \int_\Omega E \cdot \operatorname{curl}\operatorname{curl} H - H \cdot \operatorname{curl}\operatorname{curl} E \,\mathrm{d}V &= \int_\Gamma (E \times \operatorname{curl} H - H \times \operatorname{curl} E) \cdot \nu \,\mathrm{d}\sigma \\ &= \int_\Gamma (\nu \times E) \cdot \operatorname{curl} H - (\nu \times H) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \end{aligned} \tag{1}$$

Proposition 2 (Fundamental Theorem of Vector Analysis)**.**

$$\begin{aligned}E(x) = &-\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) + \nabla\int_\Gamma \nu(y)\cdot E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y)\\ &- ik\int_\Gamma \nu(y)\times H(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) + \operatorname{curl}\int_\Omega \big\{\operatorname{curl}E(y) - ikH(y)\big\}\,\Phi_k(x,y)\,\mathrm{d}V(y)\\ &- \nabla\int_\Omega \operatorname{div}E(y)\,\Phi_k(x,y)\,\mathrm{d}V(y) + ik\int_\Omega \big\{\operatorname{curl}H(y) + ikE(y)\big\}\,\Phi_k(x,y)\,\mathrm{d}V(y).\end{aligned}$$

Proposition 3 (Stratton-Chu Representation Formula)**.** If $E, H \in C^1(\Omega_+)\cap C(\Omega_+\cup\Gamma)$ satisfy Maxwell equations in Ω_+ and the Silver-Müller radiation condition, then for $x \in \Omega_+$

$$E(x) = \operatorname{curl}\int_\Gamma \nu(x)\times E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y)$$

$$H(x) = \operatorname{curl}\int_\Gamma \nu(x)\times H(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) - \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y).$$

For $x \in \Omega_-$:

$$E(x) = -\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) - \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y)$$

$$H(x) = -\operatorname{curl}\int_\Gamma \nu(y)\times H(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y)$$

Proposition 4 (Far Field Patterns)**.**

$$\begin{aligned}E^\infty(\hat{x}) &= ik\,\hat{x}\times\int_\Gamma \big\{\nu(y)\times E(y) + (\nu(y)\times H(y))\times\hat{x}\big\}e^{-ik\hat{x}\cdot y}\,\mathrm{d}\sigma(y)\\ H^\infty(\hat{x}) &= ik\,\hat{x}\times\int_\Gamma \big\{\nu(y)\times H(y) - (\nu(y)\times E(y))\times\hat{x}\big\}e^{-ik\hat{x}\cdot y}\,\mathrm{d}\sigma(y)\end{aligned}$$

Proposition 5 (Rellich Lemma)**.** If $E, H \in C^1(\Omega_+)$ is a radiating solution of Maxwell equations such that the electric far field pattern vanishes identically, then $E = H = 0$ in Ω_+.

Definition 2.

1. Γ: The regular (Lipschitzian) boundary of the open bounded set Ω_i in $\mathbb{R}^3$.
2. The tangential differentiation ∇_t is defined by

$$\nabla_\mathrm{t} := \nu\times(\nu\times\nabla).$$

3. Given a tangential vector field a, the surface divergence $\operatorname{div}_\Gamma a$ is defined as

$$\int_\Gamma \phi\,\operatorname{div}_\Gamma a\,\mathrm{d}\sigma = -\int_\Gamma \nabla_\mathrm{t}\phi\cdot a\,\mathrm{d}\sigma, \quad \forall\phi\in C^\infty(\mathbb{R}^3)$$

4. $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma) = \{v \mid v \in \mathbf{L}_2(\Gamma),\ \nu\cdot v = 0,\ \operatorname{div}_\Gamma v \in L_2(\Gamma)\}$.

5. $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) = \{v \,|\, v \in \mathbf{L}_2(\Gamma),\ \nu \cdot v = 0,\ \mathbf{curl}_\Gamma\, v \in L_2(\Gamma)\}$.

Proposition 6. $v \to \nu \times v$ is an isomorphism from $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ to $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$ with inverse $w \to -\nu \times w$, and we have

$$\begin{aligned} \mathrm{curl}_\Gamma\, v &= -\,\mathrm{div}_\Gamma(\nu \times v) \\ \mathrm{div}_\Gamma\, w &= \mathrm{curl}_\Gamma(\nu \times w) \end{aligned}$$

for $v \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma), w \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$.

Definition 3 (The Maxwell Problem)**.** The Maxwell problem is to find a pair of solution (E, H) to the Maxwell equations

$$\begin{aligned} \mathrm{curl}\, E - ikH &= 0 \\ \mathrm{curl}\, H + ikE &= 0 \end{aligned}$$

in Ω_+, with the boundary condition

$$\nu \times E|_+ = f \tag{2}$$

on Γ where $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$, and (E, H) satisfies the Silver-Müller radiation condition

$$H \times \frac{x}{|x|} - E = \mathcal{O}\left(|x|^{-2}\right) \quad |x| \to \infty. \tag{3}$$

The data to far field pattern operator $G : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is defined as

$$Gf = E^\infty \tag{4}$$

where E^∞ denotes the far field pattern of the solution E of the Maxwell problem.

2 Reciprocity Relations

Assume $x, z \in \Omega_+$, $\hat{x}, d \in \mathbb{S}^2$, $p, q \in \mathbb{R}^3$.

Given the incident electromagentic wave

$$\begin{aligned} E^{\mathrm{i}}(x, d, p) &= \frac{i}{k}\, \mathrm{curl}_x\, \mathrm{curl}_x\, p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d}, \\ H^{\mathrm{i}}(x, d, p) &= \mathrm{curl}_x\, p e^{ikx\cdot d} = ik(d \times p) e^{ikx\cdot d}, \end{aligned}$$

the scattered field is denoted by

$$E^{\mathrm{s}}(x, d, p), \quad H^{\mathrm{s}}(x, d, p)$$

with corresponding far field pattern

$$E^\infty(\hat{x}, d, p), \quad H^\infty(\hat{x}, d, p).$$

Given the incident dipole

$$\begin{aligned} E^{\mathrm{i}}_{\mathrm{p}}(x, z, p) &= \frac{i}{k}\, \mathrm{curl}_x\, \mathrm{curl}_x\, p\, \Phi_k(x, z), \\ H^{\mathrm{i}}_{\mathrm{p}}(x, z, p) &= \mathrm{curl}_x\, p\, \Phi_k(x, z), \end{aligned}$$

the scattered field is denoted by

$$E_{\mathrm{p}}^{\mathrm{s}}(x,z,p),\quad H_{\mathrm{p}}^{\mathrm{s}}(x,z,p)$$

with the corresponding far field pattern

$$E_{\mathrm{p}}^{\infty}(\hat{x},z,p),\quad H_{\mathrm{p}}^{\infty}(\hat{x},z,p).$$

The total field is denoted by

$$\begin{aligned} E(x,d,p) &= E^{\mathrm{i}}(x,d,p) + E^{\mathrm{s}}(x,d,p)\\ H(x,d,p) &= H^{\mathrm{i}}(x,d,p) + H^{\mathrm{s}}(x,d,p)\\ E_{\mathrm{p}}(x,z,p) &= E_{\mathrm{p}}^{\mathrm{i}}(x,z,p) + E_{\mathrm{p}}^{\mathrm{s}}(x,z,p)\\ H_{\mathrm{p}}(x,z,p) &= H_{\mathrm{p}}^{\mathrm{i}}(x,z,p) + H_{\mathrm{p}}^{\mathrm{s}}(x,z,p) \end{aligned}$$

Theorem 1 (Mixed Reciprocity Relation)**.**

$$p\cdot E^{\mathrm{s}}(z,-\hat{x},q) = 4\pi q\cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p)$$

Proof.

$$\begin{aligned} 4\pi q\cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p) = \int_\Gamma & \nu(y)\times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p)\cdot H^{\mathrm{i}}(y,-\hat{x},q)\\ &+ \nu(y)\times H_{\mathrm{p}}^{\mathrm{s}}(y,z,p)\cdot E^{\mathrm{i}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) \end{aligned} \tag{5}$$

From Green formula

$$\int_\Gamma \nu(y)\times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p)\cdot H^{\mathrm{s}}(y,-\hat{x},q) + \nu(y)\times H_{\mathrm{p}}^{\mathrm{s}}(y,z,p)\cdot E^{\mathrm{s}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) = 0 \tag{6}$$

Add (5), (6) and apply the boundary condition

$$\nu(y)\times E(y,-\hat{x},q) = 0 \quad y\in\Gamma$$

we have

$$4\pi q\cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p) = \int_\Gamma \nu(y)\times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p)\cdot H(y,-\hat{x},q)\,\mathrm{d}\sigma(y) \tag{7}$$

From Stratton-Chu representation,

$$\begin{aligned} E^{\mathrm{s}}(z,-\hat{x},q) = \operatorname{curl}\int_\Gamma & \nu(y)\times E^{\mathrm{s}}(y,-\hat{x},q)\,\Phi_k(z,y)\,\mathrm{d}\sigma(y)\\ &+ \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H^{\mathrm{s}}(y,-\hat{x},q)\,\Phi_k(z,y)\,\mathrm{d}\sigma(y) \end{aligned} \tag{8}$$

From Green formula

$$\begin{aligned} 0 = \operatorname{curl}\int_\Gamma & \nu(y)\times E^{\mathrm{i}}(y,-\hat{x},q)\,\Phi_k(z,y)\,\mathrm{d}\sigma(y)\\ &+ \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H^{\mathrm{i}}(y,-\hat{x},q)\,\Phi_k(z,y)\,\mathrm{d}\sigma(y) \end{aligned} \tag{9}$$

Add (8), (9) and apply the boundary condition

$$\nu(y) \times E(y, -\hat{x}, q) = 0 \quad y \in \Gamma$$

we have

$$E^{\mathrm{s}}(z, -\hat{x}, q) = \frac{i}{k} \operatorname{curl} \operatorname{curl} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \tag{10}$$

From (10), the identity

$$p \cdot \operatorname{curl} \operatorname{curl}_z \{a(y)\, \Phi_k(z, y)\} = a(y) \cdot \operatorname{curl} \operatorname{curl}_z \{p\, \Phi_k(z, y)\},$$

and the boundary condition

$$\nu(y) \times E^{\mathrm{i}}_{\mathrm{p}}(y, z, p) = -\nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \quad y \in \Gamma$$

we have

$$\begin{aligned}
p \cdot E^{\mathrm{s}}(z, -\hat{x}, q) &= \frac{i}{k}\, p \cdot \operatorname{curl} \operatorname{curl} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \\
&= \frac{i}{k} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q) \cdot \operatorname{curl} \operatorname{curl} \{p\, \Phi_k(z, y)\}\, \mathrm{d}\sigma(y) \\
&= \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q) \cdot E^{\mathrm{i}}_{\mathrm{p}}(y, z, p)\, \mathrm{d}\sigma(y) \\
&= -\int_{\Gamma} \nu(y) \times E^{\mathrm{i}}_{\mathrm{p}}(y, z, p) \cdot H(y, -\hat{x}, q)\, \mathrm{d}\sigma(y) \\
&= \int_{\Gamma} \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \cdot H(y, -\hat{x}, q)\, \mathrm{d}\sigma(y),
\end{aligned}$$

which equals (7). □

Theorem 2 (Reciprocity Relation)**.**

$$q \cdot E^{\infty}(\hat{x}, d, p) = p \cdot E^{\infty}(-d, -\hat{x}, q)$$

Proof. Apply Green formula to E^{i} in Ω_-, E^{s} in Ω_+, we have

$$\int_{\Gamma} \big\{\nu(y) \times E^{\mathrm{i}}(y, d, p) \cdot H^{\mathrm{i}}(y, -\hat{x}, q) - \nu(y) \times E^{\mathrm{i}}(y, -\hat{x}, q) \cdot H^{\mathrm{i}}(y, d, p)\big\}\, \mathrm{d}\sigma(y) = 0 \tag{11}$$

$$\int_{\Gamma} \big\{\nu(y) \times E^{\mathrm{s}}(y, d, p) \cdot H^{\mathrm{s}}(y, -\hat{x}, q) - \nu(y) \times E^{\mathrm{s}}(y, -\hat{x}, q) \cdot H^{\mathrm{s}}(y, d, p)\big\}\, \mathrm{d}\sigma(y) = 0 \tag{12}$$

$$\begin{aligned}
4\pi q \cdot E^{\infty}(\hat{x}, d, p) = \int_{\Gamma} \big\{&\nu(y) \times E^{\mathrm{s}}(y, d, p) \cdot H^{\mathrm{i}}(y, -\hat{x}, q) \\
&+ \nu(y) \times H^{\mathrm{s}}(y, d, p) \cdot E^{\mathrm{i}}(y, -\hat{x}, q)\big\}\, \mathrm{d}\sigma(y)
\end{aligned} \tag{13}$$

Interchange p, q and $d, \hat{x}$ respectively in (13), we have

$$\begin{aligned}
4\pi q \cdot E^{\infty}(\hat{x}, d, p) = \int_{\Gamma} \big\{&\nu(y) \times E^{\mathrm{s}}(y, -\hat{x}, q) \cdot H^{\mathrm{i}}(y, d, p) \\
&+ \nu(y) \times H^{\mathrm{s}}(y, -\hat{x}, q) \cdot E^{\mathrm{i}}(y, d, p)\big\}\, \mathrm{d}\sigma(y)
\end{aligned} \tag{14}$$

Subtract (13) with (14) and add (11), (12), together with the boundary condition

$$\nu(y) \times E(y, d, p) = \nu(y) \times E(y, -\hat{x}, p) = 0, \quad y \in \Gamma$$

the result follows. □

3 The Uniqueness Theorem

Theorem 3. If D_1 and D_2 are two perfect conductors such that the electric far field patterns coincide for a fixed wave number, all incident directions and all observation directions, then $D_1 = D_2$.

Proof. Let U be the unbounded component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. By Rellich lemma,

$$E_1^{\mathrm{s}}(x, d, p) = E_2^{\mathrm{s}}(x, d, p) \quad \forall x \in U, d, p \in \mathbb{S}^2.$$

By mixed reciprocity relation,

$$E_1^{\infty}(\hat{x}, z, p) = E_2^{\infty}(\hat{x}, z, p) \quad \forall z \in U, \hat{x}, p \in \mathbb{S}^2.$$

Again by Rellich lemma,

$$E_{\mathrm{p},1}^{\mathrm{s}}(x, z, p) = E_{\mathrm{p},2}^{\mathrm{s}}(x, z, p) \quad \forall x, z \in U, p \in \mathbb{S}^2.$$

Assume $D_1 \neq D_2$, then $\exists\, \widetilde{x} \in U$ such that $\widetilde{x} \in \partial D_1, \widetilde{x} \notin \overline{D_2}$. Construct $\{z_n\}$ such that $z_n = \widetilde{x} + \frac{1}{n}\nu(\widetilde{x}) \in U$ for sufficiently large n. From the well-posedness of the solution on D_2, $E_{\mathrm{p},2}^{\mathrm{s}}(\widetilde{x}, \widetilde{x}, p)$ is well-behaved. But

$$E_{\mathrm{p},1}^{\mathrm{s}}(\widetilde{x}, z_n, q) \to \infty \text{ as } z_n \to \widetilde{x} \text{ and given } p \perp \nu(\widetilde{x})$$

in order to fulfill the boundary condition with the incident dipole $E_{\mathrm{p},1}^{\mathrm{i}}(\widetilde{x}, z_n, p)$, which becomes unbounded as $z_n \to \widetilde{x}$. □

4 The Factorization Method

Definition 4 (The Far Field Operator)**.** The far field operator $F : \mathbf{L}_{\mathrm{t}}^2(\mathbb{S}^2) \to \mathbf{L}_{\mathrm{t}}^2(\mathbb{S}^2)$ is

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} E^{\infty}(\hat{x}, \theta, g(\theta))\, \mathrm{d}\sigma(\theta), \quad \hat{x} \in \mathbb{S}^2. \tag{15}$$

Proposition 7. The far field operator F is normal, i.e. $F^*F = FF^*$.

Proof. Let $g, h \in \mathbf{L}_{\mathrm{t}}^2(\mathbb{S}^2)$ and define the Herglotz wave functions $v^{\mathrm{i}}, w^{\mathrm{i}}$ with density g, h respectively:

$$v^{\mathrm{i}}(x) = \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta), \quad x \in \mathbb{R}^3$$
$$w^{\mathrm{i}}(x) = \int_{\mathbb{S}^2} h(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta). \quad x \in \mathbb{R}^3$$

Let v, w be solutions of the scattering problem corresponding to incident fields $v^{\mathrm{i}}, w^{\mathrm{i}}$ with scattered fields $v^{\mathrm{s}} = v - v^{\mathrm{i}}, w^{\mathrm{s}} = w - w^{\mathrm{i}}$ and far field patterns v^{∞}, w^{∞} respectively. Apply Green theorem in $\Omega_R = \{x \in \mathbb{R}^3 \setminus \overline{\Omega} : |x| < R\}$ with sufficiently big R, together with the boundary condition we have

$$0 = \int_{\Omega_R} \{v\, \Delta\overline{w} - \overline{w}\, \Delta v\}\ \mathrm{d}V \tag{16}$$
$$= \int_{\mathbb{S}^2} \{\overline{w} \times \operatorname{curl} v - v \times \operatorname{curl} \overline{w}\} \cdot \nu\, \mathrm{d}\sigma. \tag{17}$$

Decomposing $v = v^{\mathrm{i}} + v^{\mathrm{s}}$ and $w = w^{\mathrm{i}} + w^{\mathrm{s}}$, we split (17) into the sum of the following four parts:

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \cdot \nu \, \mathrm{d}\sigma, \tag{18}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \nu \, \mathrm{d}\sigma, \tag{19}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \cdot \nu \, \mathrm{d}\sigma, \tag{20}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \nu \, \mathrm{d}\sigma. \tag{21}$$

The integral (18) vanishes by applying Green theorem in $B_R = \{x : |x| < R\}$. To evaluate the integral (19), we note by the radiation condition

$$\overline{w^{\mathrm{s}}} \times \hat{x} - \frac{1}{ik} \operatorname{curl} \overline{w^{\mathrm{s}}} = \mathcal{O}\left(r^{-2}\right) \tag{22}$$

$$v^{\mathrm{s}} \times \hat{x} + \frac{1}{ik} \operatorname{curl} v^{\mathrm{s}} = \mathcal{O}\left(r^{-2}\right) \tag{23}$$

and relations between scattered fields and far field patterns

$$\overline{w^{\mathrm{s}}} = \frac{e^{-ikr}}{4\pi r} \left\{ \overline{w^{\infty}} + \mathcal{O}\left(r^{-1}\right) \right\}$$
$$v^{\mathrm{s}} = \frac{e^{ikr}}{4\pi r} \left\{ v^{\infty} + \mathcal{O}\left(r^{-1}\right) \right\}$$

one obtains

$$\begin{aligned}
&\left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \hat{x} \\
&= ik \left\{ \overline{w^{\mathrm{s}}} \times (\hat{x} \times v^{\mathrm{s}}) + v^{\mathrm{s}} \times (\hat{x} \times \overline{w^{\mathrm{s}}}) \right\} \cdot \hat{x} \\
&= 2ik \left\{ \overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} - (\overline{w^{\mathrm{s}}} \cdot \hat{x}) (v^{\mathrm{s}} \cdot \hat{x}) \right\} \\
&= 2ik \, \overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} \\
&= \frac{ik}{8\pi^2 r^2} \overline{w^{\infty}} \cdot v^{\infty} + \mathcal{O}\left(r^{-3}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \nu \, \mathrm{d}\sigma \\
&\longrightarrow \frac{ik}{8\pi^2} \int_{\mathbb{S}^2} \overline{w^{\infty}} \cdot v^{\infty} \, \mathrm{d}\sigma = \frac{ik}{8\pi^2} (Fg, Fh)_{L^2(\mathbb{S}^2)}
\end{aligned}$$

To evaluate the integral (20), one note that it can be rearranged as

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \cdot \nu \, \mathrm{d}\sigma \tag{24}$$

$$= - \int_{\mathbb{S}^2} \left\{ (\hat{x} \times \operatorname{curl} v^{\mathrm{s}}) \cdot \overline{w^{\mathrm{i}}} + (\hat{x} \times v^{\mathrm{s}}) \cdot \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \mathrm{d}\sigma \tag{25}$$

Substitute

$$\overline{w^{\mathrm{i}}}(x) = \int_{\mathbb{S}^2} h(\theta)\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta),$$
$$\operatorname{curl} \overline{w^{\mathrm{i}}}(x) = ik \int_{\mathbb{S}^2} (h(\theta)\times\theta)\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)$$

into (25), it becomes

$$-\int_{|x|=r} (\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\cdot\left\{\int_{\mathbb{S}^2} h(\theta)\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\right\} \mathrm{d}\sigma(x)$$
$$-\int_{|x|=r} (\hat{x}\times v^{\mathrm{s}})\cdot\left\{ik \int_{\mathbb{S}^2} (h(\theta)\times\theta)\ e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\right\} \mathrm{d}\sigma(x). \quad (26)$$

From $h(\theta)\cdot\theta = 0$ and $\theta\cdot\theta = 1$, by formulae

$$a\times(b\times c) = b\ (a\cdot c) - c\ (a\cdot b)$$
$$a\cdot(b\times c) = -b\cdot(a\times c)$$

we have

$$h(\theta)\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) = h(\theta)\cdot\{(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) - \theta\,(\theta\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}))\}$$
$$= h(\theta)\cdot\{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta)\}$$

and

$$(\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta) = h(\theta)\cdot(\theta\times(\hat{x}\times v^{\mathrm{s}}))$$

Substitute into (26), the integral (20) is

$$-\int_{\mathbb{S}^2}\int_{|x|=r} \big\{h(\theta)\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) + ik\ (\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta)\big\}\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta)$$
$$= -\int_{\mathbb{S}^2} h(\theta)\cdot\left\{\int_{|x|=r} \big\{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta) + ik\,\theta\times(\hat{x}\times v^{\mathrm{s}})\big\}\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right\} \mathrm{d}\sigma(\theta)$$
$$\longrightarrow -\,(Fg,h)_{L^2(\mathbb{S}^2)}\,.$$

By the same token, the integral (21) is $(g,Fh)_{L^2(\mathbb{S}^2)}$. Hence

$$0 = (g,Fh)_{L^2(\mathbb{S}^2)} - (Fg,h)_{L^2(\mathbb{S}^2)} + \frac{ik}{8\pi^2}\,(Fg,Fh)_{L^2(\mathbb{S}^2)}\,,$$

the identity

$$F - F^* = \frac{ik}{8\pi^2}F^*F$$

follows.

Now set $S = I + \frac{ik}{8\pi^2}F$, we have

$$S^*S = \left(I - \frac{ik}{8\pi^2}F^*\right)\left(I + \frac{ik}{8\pi^2}F\right)$$
$$= I + \frac{ik}{8\pi^2}F - \frac{ik}{8\pi^2}F^* + \frac{k^2}{64\pi^2}F^*F$$
$$= I.$$

If $Sg = 0$, then $g = S^*Sg = 0$, hence S is injective. Note that S is a compact perturbation of the identity, from Fredholm theory S is an isomorphism. Therefore $S^* = S^{-1}$ and $SS^* = I$. Comparing S^*S and SS^* we can see that $F^*F = FF^*$, i.e. F is normal. □

Proposition 8. The data to far field pattern operator G is compact, injective with dense range.

Proof. (c.f. Cakoni et al. [3, Theorem 3.1], Cakoni et al. [4, Lemma 3.8])

$$Gf = \frac{ik}{4\pi}\hat{x} \times \int_\Gamma \left\{ \nu(y) \times E(y) + \frac{1}{ik}\big(\nu(y) \times \operatorname{curl} E(y)\big) \times \hat{x} \right\} e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(y)$$

$$\begin{aligned}\langle Gf, g\rangle = \frac{ik}{4\pi}\int_\Gamma \int_{\mathbb{S}^2} \Big\{ &\big(\hat{x} \times (\nu(y) \times E(y))\big) \cdot g(\hat{x}) \\ &+ \frac{1}{ik}\Big(\hat{x} \times \big(\nu(y) \times \operatorname{curl} E(y)\big) \times \hat{x}\Big) \cdot g(\hat{x}) \Big\} e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(\hat{x})\, \mathrm{d}\sigma(y) \end{aligned} \tag{27}$$

$$\begin{aligned} \big(\hat{x} \times (\nu(y) \times E(y))\big) \cdot g(\hat{x}) &= (\nu(y) \times E(y)) \cdot (g(\hat{x}) \times \hat{x}) \\ \Big(\hat{x} \times \big(\nu(y) \times \operatorname{curl} E(y)\big) \times \hat{x}\Big) \cdot g(\hat{x}) &= \big(\nu(y) \times \operatorname{curl} E(y)\big) \cdot \big(\hat{x} \times (g(\hat{x}) \times \hat{x})\big) \end{aligned}$$

Note that $g(\hat{x})$ is tangential on $\mathbb{S}^2$ (i.e. $g(\hat{x}) \cdot \hat{x} = 0$), then $\hat{x} \times (g(\hat{x}) \times \hat{x}) = g(\hat{x})$; (27) becomes

$$\begin{aligned}\langle Gf, g\rangle = \frac{ik}{4\pi}\int_\Gamma \int_{\mathbb{S}^2} \Big\{ &(\nu(y) \times E(y)) \cdot (g(\hat{x}) \times \hat{x}) \\ &+ \frac{1}{ik}\big(\nu(y) \times \operatorname{curl} E(y)\big) \cdot g(\hat{x}) \Big\} e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(\hat{x})\, \mathrm{d}\sigma(y) \end{aligned} \tag{28}$$

Define

$$E_g(y) = \int_{\mathbb{S}^2} g(\hat{x}) e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(\hat{x})$$

Then

$$\begin{aligned} \operatorname{curl} E_g(y) &= ik \int_{\mathbb{S}^2} (g(\hat{x}) \times \hat{x})\, e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(\hat{x}) \\ \operatorname{curl}\operatorname{curl} E_g(y) &= k^2 \int_{\mathbb{S}^2} (\hat{x} \times (g(\hat{x}) \times \hat{x}))\, e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(\hat{x}) \end{aligned}$$

For $g(\hat{x})$ is tangential,

$$\operatorname{curl}\operatorname{curl} E_g(y) = k^2 E_g(y).$$

Now (28) can be further simplified to

$$\langle Gf, g\rangle = \frac{1}{4\pi}\int_\Gamma \Big\{ (\nu(y) \times E(y)) \cdot \operatorname{curl} E_g(y) + \big(\nu(y) \times \operatorname{curl} E(y)\big) \cdot E_g(y) \Big\}\, \mathrm{d}\sigma(y) \tag{29}$$

where $\widetilde{E}$ is the solution of

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\widetilde{E} - k^2\widetilde{E} &= 0 \quad \text{in } \Omega_+ \\ \nu \times \big(\widetilde{E} - E_g\big) &= 0 \quad \text{on } \Gamma \end{aligned}$$

$$\langle Gf, g\rangle = \frac{1}{4\pi}\int_\Gamma \big(\nu(y) \times E(y)\big) \cdot \big(\operatorname{curl} E_g(y) - \operatorname{curl}\widetilde{E}(y)\big)\,\mathrm{d}\sigma(y) \tag{30}$$

$$\langle Gf, g\rangle = \frac{1}{4\pi}\int_\Gamma f(y) \cdot \big(\operatorname{curl} E_g(y) - \operatorname{curl}\widetilde{E}(y)\big)\,\mathrm{d}\sigma(y) \tag{31}$$

Hence,

$$(G^* g)(y) = 4\pi\nu(y) \times \big(\operatorname{curl} E_g(y) - \operatorname{curl}\widetilde{E}(y)\big) \times \nu(y), \quad y \in \Gamma. \tag{32}$$

Let $G^* g = 0$, then $\nu \times (\operatorname{curl} E_g - \operatorname{curl}\widetilde{E}) = 0$ on Γ; we already have $\nu \times (E_g - \widetilde{E}) = 0$ on Γ. Now let $B_R = \{x | \|x\| < R\}$ be a ball with radius R that enclose $\overline{\Omega}$. □

Proposition 9. For $z \in \mathbb{R}^3$ and a fixed $d \in \mathbb{S}^2$, define

$$\varphi_z(\hat{x}) = ik\,(\hat{x} \times d)\,e^{ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

then φ_z belongs to the range of G iff $z \in \Omega$.

Proof. Assume first $z \in \Omega$. define

$$v(x) = \operatorname{curl}_x \{\Phi_k(x, z)\,d\}, \quad \forall x \in \mathbb{R}^3 \setminus \Omega$$

and $f = v|_\Gamma$. The far field pattern of v, denoted by v^∞, is

$$v^\infty(\hat{x}) = ik\,(\hat{x} \times d)\,e^{ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

which is identical to φ_z. From $Gf = v^\infty = \varphi_z$, φ_z belongs to the range of G.

Now assume $z \notin \Omega$ and there exists f with $Gf = \varphi_z$. Let v be the radiating solution of the Maxwell problem with boundary data f and $v^\infty = Gf$ be the far field pattern of v. Note that the far field pattern of $\operatorname{curl}\{\Phi_k(\cdot, z)\,d\}$ is φ_z, from Rellich lemma $v(x) = \operatorname{curl}\{\Phi_k(x, z)\,d\}$ for all x outside of any sphere which contains both z and Ω. By analytic continuation, v and $\operatorname{curl}\{\Phi_k(\cdot, z)\,d\}$ coincide on $\mathbb{R}^3 \setminus \big(\overline{\Omega} \cup \{z\}\big)$. But if $z \notin \overline{\Omega}$, then $\operatorname{curl}\{\Phi_k(x, z)\,d\}$ is singular on $x = z$, while v is analytic on $\mathbb{R}^3 \setminus \overline{\Omega}$, a contradiction. Otherwise if $z \in \Gamma$, then $x \mapsto \operatorname{curl}\{\Phi_k(x, z)\,d\}$ for $x \in \Gamma, x \neq z$, is in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. But $\operatorname{curl}\{\Phi(x, z)\,d\}$ does not belong to $\mathbf{H}_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \Omega)$ or $\mathbf{H}(\operatorname{curl}, \Omega)$, for $\operatorname{curl}\Phi_k(x, z) = \mathcal{O}\left(|x - z|^{-2}\right)$ if $x \to z$. □

Definition 5. The single layer operator $S_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ with density f is

$$(S_k f)(x) = \int_\Gamma f(y)\,\Phi_k(x, y)\,\mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{33}$$

The vector single layer operator $S_k : \mathbf{H}^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is formed with vector density g:

$$(S_k g)(x) = \int_\Gamma g(y)\,\Phi_k(x, y)\,\mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{34}$$

The electric dipole operator $N_k : \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$ is

$$(N_k f)(x) = \nu(x) \times \operatorname{curl}\operatorname{curl}_x \int_\Gamma \big(\nu(y) \times f(y)\big)\Phi_k(x, y)\,\mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{35}$$

By $\operatorname{curl}\operatorname{curl}\cdot = \nabla \operatorname{div}\cdot + \Delta\cdot$,

$$\begin{aligned} N_k f &= \nu \times \operatorname{curl}\operatorname{curl} S_k(\nu \times f) \\ &= k^2 \nu \times S_k(\nu \times f) + \nu \times \nabla S_k(\operatorname{div}_\Gamma(\nu \times f)) \end{aligned} \tag{36}$$

We note the following formula: for scalar f, vector g

$$\int_\Gamma \langle \nu \times \nabla f, g\rangle = -\int_\Gamma f\langle \nu, \operatorname{curl} g\rangle$$

This can be verified with

$$\int_\Omega \operatorname{curl} u = \int_\Gamma \nu \times u$$

and the proof runs as follows:

$$\begin{aligned} \int_\Gamma \langle \nu \times \nabla f, g\rangle &= -\int_\Gamma \langle g \times \nabla f, \nu\rangle = -\int_\Omega \operatorname{div}(g \times \nabla f) \\ &= -\int_\Omega \langle \operatorname{curl} g, \nabla f\rangle \\ &= -\int_\Omega \operatorname{div}(f \operatorname{curl} g) = -\int_\Gamma f\langle \nu, \operatorname{curl} g\rangle \end{aligned}$$

Set $f = S_k(\operatorname{div}_\Gamma(\nu \times \varphi))$, $g = \overline{\psi}$ and recall that $\operatorname{div}_\Gamma(\nu \times \overline{\psi}) = -\nu \cdot \operatorname{curl}\overline{\psi}$, we have

$$\begin{aligned} \langle N_k\varphi,\, \psi\rangle &= \langle k^2\nu \times S_k(\nu \times \varphi) + \nu \times \nabla S_k(\operatorname{div}_\Gamma \nu \times \varphi),\, \psi\rangle \\ &= k^2 \int_\Gamma (\nu \times S_k(\nu \times \varphi)) \cdot \overline{\psi} + \int_\Gamma (\nu \times \nabla S_k(\operatorname{div}_\Gamma(\nu \times \varphi))) \cdot \overline{\psi} \\ &= -k^2 \int_\Gamma S_k(\nu \times \varphi) \cdot \left(\nu \times \overline{\psi}\right) + \int_\Gamma S_k(\operatorname{div}_\Gamma(\nu \times \varphi)) \left(\nu \cdot \operatorname{curl}\overline{\psi}\right) \\ &= -k^2 \int_\Gamma S_k(\nu \times \varphi) \cdot \overline{(\nu \times \psi)} + \int_\Gamma S_k(\operatorname{div}_\Gamma(\nu \times \varphi)) \overline{\operatorname{div}_\Gamma(\nu \times \psi)} \\ &= -k^2 \langle S_k(\nu \times \varphi),\, \nu \times \psi\rangle + \langle S_k(\operatorname{div}_\Gamma(\nu \times \varphi)),\, \operatorname{div}_\Gamma(\nu \times \psi)\rangle\,. \end{aligned} \tag{37}$$

Proposition 10. The adjoint operator $N_k^* : \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$ is N_{-k}, i.e.

$$(N_k^* f)(x) = \nu(x) \times \operatorname{curl}_x \operatorname{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi_{-k}(x,y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{38}$$

Proof. Note that

$$\begin{gathered} \nabla_x \cdot \nabla_y \Phi_{-k}(x,y) = -\nabla_y \cdot \nabla_y \Phi_{-k}(x,y), \\ \left(\left(\nu(y) \times \overline{g(y)}\right) \cdot \nabla_x\right) \nabla_y \Phi_{-k}(x,y) = -\left(\left(\nu(y) \times \overline{g(y)}\right) \cdot \nabla_y\right) \nabla_y \Phi_{-k}(x,y), \end{gathered}$$

which can be verified by straightforward differentiation. Then

$$
\begin{aligned}
\langle f, N_k g\rangle &= \int_\Gamma f(x)\cdot \overline{\left\{\nu(x)\times \operatorname{curl}_x \operatorname{curl}_x \int_\Gamma (\nu(y)\times g(y))\ \Phi_k(x,y)\,\mathrm{d}\sigma(y)\right\}}\,\mathrm{d}\sigma(x)\\
&= \int_\Gamma\int_\Gamma f(x)\cdot\left\{\nu(x)\times \operatorname{curl}_x\left(\nabla_x\Phi_{-k}(x,y)\times\left(\nu(y)\times\overline{g(y)}\right)\right)\right\}\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x)\\
&= \int_\Gamma\int_\Gamma f(x)\cdot\left\{\nu(x)\times \operatorname{curl}_x\left(\nabla_x\Phi_{-k}(x,y)\times\left(\nu(y)\times\overline{g(y)}\right)\right)\right\}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\int_\Gamma (f(x)\times\nu(x))\cdot \operatorname{curl}_x\left(\nabla_x\Phi_{-k}(x,y)\times\left(\nu(y)\times\overline{g(y)}\right)\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\int_\Gamma (\nu(x)\times f(x))\cdot \operatorname{curl}_x\left(\nabla_y\Phi_{-k}(x,y)\times\left(\nu(y)\times\overline{g(y)}\right)\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\int_\Gamma (\nu(x)\times f(x))\cdot\Big\{-\left(\nu(y)\times\overline{g(y)}\right)(\nabla_x\cdot\nabla_y\Phi_{-k}(x,y))\\
&\qquad\qquad + \left(\left(\nu(y)\times\overline{g(y)}\right)\cdot\nabla_x\right)\nabla_y\Phi_{-k}(x,y)\Big\}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\int_\Gamma (\nu(x)\times f(x))\cdot\Big\{\left(\nu(y)\times\overline{g(y)}\right)(\nabla_y\cdot\nabla_y\Phi_{-k}(x,y))\\
&\qquad\qquad - \left(\left(\nu(y)\times\overline{g(y)}\right)\cdot\nabla_y\right)\nabla_y\Phi_{-k}(x,y)\Big\}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\left\{-\operatorname{curl}_y\operatorname{curl}_y\int_\Gamma(\nu(x)\times f(x))\,\Phi_{-k}(x,y)\,\mathrm{d}\sigma(x)\right\}\cdot\left(\nu(y)\times\overline{g(y)}\right)\,\mathrm{d}\sigma(y)\\
&= \int_\Gamma\left\{\nu(y)\times\operatorname{curl}_y\operatorname{curl}_y\int_\Gamma(\nu(x)\times f(x))\,\Phi_{-k}(x,y)\,\mathrm{d}\sigma(x)\right\}\cdot\overline{g(y)}\,\mathrm{d}\sigma(y)\\
&= \langle N_k^* f, g\rangle.
\end{aligned}
$$

□

Proposition 11.

$$F = \frac{1}{k^2} G N_{-k} G^*.$$

Proof. Define auxiliary operator $\mathcal{H} : \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2) \to \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$ as

$$(\mathcal{H}g)(x) = \nu(x)\times\int_{\mathbb{S}^2} g(\theta)\,e^{ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\quad x\in\Gamma,$$

then the adjoint operator $\mathcal{H}^* : \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is

$$(\mathcal{H}^* f)(\theta) = \theta\times\left(\theta\times\int_\Gamma(\nu(x)\times f(x))\,e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right),\quad \theta\in\mathbb{S}^2. \tag{39}$$

This can be verified by

$$
\begin{aligned}
\langle f, \mathcal{H}g\rangle &= \int_\Gamma f(x)\cdot \overline{\left\{\nu(x)\times\int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\right\}}\,\mathrm{d}\sigma(x)\\
&= \int_\Gamma\int_{\mathbb{S}^2} f(x)\cdot\left(\nu(x)\times\overline{g(\theta)}\right) e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\,\mathrm{d}\sigma(x)\\
&= \int_{\mathbb{S}^2}\int_\Gamma f(x)\cdot\left(\nu(x)\times\overline{g(\theta)}\right) e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma (f(x)\times\nu(x))\cdot\overline{g(\theta)}\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma (f(x)\times\nu(x))\cdot\left(\left(\theta\times\overline{g(\theta)}\right)\times\theta\right) e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma (\theta\times(f(x)\times\nu(x)))\cdot\left(\theta\times\overline{g(\theta)}\right) e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma (\theta\times((f(x)\times\nu(x))\times\theta))\cdot\overline{g(\theta)}\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\left\{\left(\theta\times\int_\Gamma (f(x)\times\nu(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right)\times\theta\right\}\cdot\overline{g(\theta)}\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\left\{\theta\times\left(\theta\times\int_\Gamma (\nu(x)\times f(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right)\right\}\cdot\overline{g(\theta)}\,\mathrm{d}\sigma(\theta)\\
&= \langle\mathcal{H}^* f, g\rangle.
\end{aligned}
$$

Given $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define $u(x)$ by

$$
u(x) = \mathrm{curl}\,\mathrm{curl}_x \int_\Gamma (\nu(y)\times f(y))\,\Phi_k(x,y)\,\mathrm{d}\sigma(y), \quad x\in\mathbb{R}^3\setminus\Gamma.
$$

From the asymptotic relation (c.f. Colton and Kress [5] (6.27))

$$
\mathrm{curl}\,\mathrm{curl}_x\left\{a(y)\frac{e^{ik|x-y|}}{|x-y|}\right\} = -k^2\frac{e^{ik|x|}}{|x|}\left\{\hat{x}\times\left(\hat{x}\times a(y)\, e^{-ik\hat{x}\cdot y}\right) + \mathcal{O}\left(|x|^{-1}\right)\right\}
$$

the far field pattern of u can be seen as $-k^2\mathcal{H}^* f$; the trace $\nu(x)\times u(x) = N_k f$. Hence, $-k^2\mathcal{H}^* f = GN_k f \implies \mathcal{H}^* = -\frac{1}{k^2}GN_k$, so $\mathcal{H} = -\frac{1}{k^2}N_k^* G^* = -\frac{1}{k^2}N_{-k}G^*$. By definition $F = -G\mathcal{H}$, hence

$$
F = -G\mathcal{H} = -G\left(-\frac{1}{k^2}N_{-k}G^*\right) = \frac{1}{k^2}GN_{-k}G^*. \tag{40}
$$

□

Proposition 12. $\Im\langle S_{-k}\varphi, \varphi\rangle < 0$ for $\varphi\in\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\varphi\neq 0$.

Proof. Given $\varphi\in\mathbf{H}^{-\frac{1}{2}}(\Gamma)$, define

$$
v(x) = \int_\Gamma \varphi(y)\,\Phi_{-k}(x,y)\,\mathrm{d}\sigma(y), \quad x\in\mathbb{R}^3\setminus\Gamma. \tag{41}
$$

Note that $\Delta v + k^2 v = 0$ for $x\in\mathbb{R}^3\setminus\Gamma$,

$$
\frac{\partial v_\pm}{\partial\nu} = \int_\Gamma \varphi(y)\left(\nabla_x\Phi_{-k}(x,y)\cdot\nu(x)\right)\,\mathrm{d}\sigma(y) \mp \frac{1}{2}\varphi(x),
$$

and v satisfies the radiation condition

$$\frac{\partial v(x)}{\partial \nu} + ikv(x) = \mathcal{O}\left(|x|^{-2}\right), \quad |x| \to \infty. \tag{42}$$

Then

$$\begin{aligned}
\langle S_{-k}\varphi,\, \varphi\rangle &= \left\langle v, \frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu} \right\rangle \\
&= \int_\Gamma v \cdot \frac{\partial \overline{v}_-}{\partial \nu} \,\mathrm{d}\sigma - \int_\Gamma v \cdot \frac{\partial \overline{v}_+}{\partial \nu} \,\mathrm{d}\sigma \\
&= \int_{B_R \cup \Omega_-} \left\{ |\nabla v|^2 - k^2 |v|^2 \right\} \mathrm{d}V - \int_{\mathbb{S}^2} v \cdot \frac{\partial \overline{v}}{\partial \nu} \,\mathrm{d}\sigma \qquad (43) \\
&= \int_{B_R \cup \Omega_-} \left\{ |\nabla v|^2 - k^2 |v|^2 \right\} \mathrm{d}V - ik \int_{\mathbb{S}^2} |v|^2 \,\mathrm{d}\sigma + \mathcal{O}\left(|x|^{-1}\right) \qquad (44)
\end{aligned}$$

where we use the radiation condition (42) into the second integral of (43). Now take the imaginary part and let $R \to \infty$,

$$\Im\langle S_{-k}\varphi,\, \varphi\rangle = -k \lim_{R\to\infty} \int_{\mathbb{S}^2} |v|^2 \,\mathrm{d}\sigma = -\frac{k}{16\pi^2} \int_{\mathbb{S}^2} \mathrm{d}\sigma(\theta) |v^\infty|^2 \leqslant 0.$$

Let $\Im\langle S_{-k}\varphi,\, \varphi\rangle = 0$ for some $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, then by (44) $v^\infty = 0$; via Rellich's lemma and unique continuation $v = 0$ in Ω_+, hence $S_{-k}\varphi = 0 \implies \varphi = 0$, for S_{-k} is an isomorphism. $\square$

Proposition 13. $\Im\langle \varphi, N_k \varphi\rangle > 0$ for $k > 0$ and $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$.

Proof. Given $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define

$$v(x) = \mathrm{curl} \int_\Gamma \nu(y) \times \varphi(y)\, \Phi(x, y) \,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma. \tag{45}$$

Note that $\mathrm{div}\, v \equiv 0$ for $x \in \mathbb{R}^3$, $\Delta v + k^2 v = 0$ for $x \in \mathbb{R}^3 \setminus \Gamma$,

$$\begin{aligned}
v_\pm(x) &= \int_\Gamma \nabla_x \Phi(x, y) \times \big(\nu(y) \times \varphi(y)\big) \,\mathrm{d}\sigma(y) \mp \frac{1}{2} \nu(x) \times \big(\nu(x) \times \varphi(x)\big) \\
&= \int_\Gamma \nabla_x \Phi(x, y) \times \big(\nu(y) \times \varphi(y)\big) \,\mathrm{d}\sigma(y) \pm \frac{1}{2} \varphi(x)
\end{aligned}$$

(c.f. Colton and Kress [5] Theorem 6.13), and the radiation condition

$$\mathrm{curl}\, v(x) \times \frac{x}{|x|} - ikv(x) = \mathcal{O}\left(|x|^{-2}\right), \quad |x| \to \infty. \tag{46}$$

By vector Green formula

$$\int_\Omega a \cdot \Delta b + \mathrm{curl}\, a \cdot \mathrm{curl}\, b + \mathrm{div}\, a \,\mathrm{div}\, b = \int_\Gamma -a \cdot (\nu \times \mathrm{curl}\, b) + (\nu \cdot a) \,\mathrm{div}\, b$$

with $a = v_\pm, b = \overline{v}$, we have

$$\begin{aligned}
\langle \varphi, N_k \varphi \rangle &= \langle v_+ - v_-, \, \nu \times \operatorname{curl} v \rangle \\
&= \int_\Gamma (v_+ - v_-) \cdot (\nu \times \operatorname{curl} \overline{v}) \, \mathrm{d}\sigma \\
&= \int_\Gamma v_+ \cdot (\nu \times \operatorname{curl} \overline{v}) \, \mathrm{d}\sigma - \int_\Gamma v_- \cdot (\nu \times \operatorname{curl} \overline{v}) \, \mathrm{d}\sigma \\
&= \int_{B_R \cup \Omega_-} \left\{ |\operatorname{curl} v|^2 - k^2 |v|^2 \right\} \mathrm{d}V + \int_{\mathbb{S}^2} v \cdot (\hat{x} \times \operatorname{curl} \overline{v}) \, \mathrm{d}\sigma \qquad (47) \\
&= \int_{B_R \cup \Omega_-} \left\{ |\operatorname{curl} v|^2 - k^2 |v|^2 \right\} \mathrm{d}V + ik \int_{\mathbb{S}^2} |v|^2 \, \mathrm{d}\sigma + \mathcal{O}\left(|x|^{-1} \right) \qquad (48)
\end{aligned}$$

where we use the radiation condition (46) into the second integral of (47). Now take the imaginary part and let $R \to \infty$,

$$\Im \langle \varphi, N_k \varphi \rangle = k \lim_{R \to \infty} \int_{\mathbb{S}^2} |v|^2 \, \mathrm{d}\sigma = \frac{k}{16\pi^2} \int_{\mathbb{S}^2} \mathrm{d}\sigma(\theta) |v^\infty|^2 \geqslant 0.$$

Let $\Im \langle \varphi, N_k \varphi \rangle = 0$ for some $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)$, then by (48) $v^\infty = 0$; via Rellich's lemma and unique continuation $v = 0$ in Ω_+, hence $N_k \varphi = 0 \Longrightarrow \varphi = 0$, for N_k is an isomorphism. □

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