1. Introduction

The recent financial crisis has caused great instability and uncertainty to the global economy, making the insurance industry more vulnerable to systemic risk. There are two main strands of thought that would provide feasible solutions to enhance the overall protection of policyholders, namely guarantees designed to provide monetary reimbursement in case of failures, and setting up an effective regulatory system for the prevention of such insolvency in advance. In our paper we emphasize the notion of effective regulation and study the quantitative impact of regulatory schemes to insurers without regards to mortality and longevity risks.

One of the simplest and most intuitive way of implementing supervision is the early warning system: when the assets of the insurer drop to a certain level above the bankruptcy threshold, the supervisory authority will step in and require the insurer to make appropriate changes to reduce their default probability; under such monitoring system each contingent regulatory scheme affects the benefits of the insured differently. Here we focus on the comparison between various regulatory schemes by examining the typical equity-linked insurance products with their payment conditions and computing the corresponding terminal expected utility of the insured under the early warning system and the single-period expected utility framework with contraints. Regulatory schemes discussed in this study include risky asset weight adjustment, capital injection, and a combination of both.

The adoption of the maximization of the single-period terminal expected utility framework to investigate the issue of insurers' asset-liability management of insurance products with guarantees began with Jensen and Sørensen (2001) and Consiglio et al. (2006). Jensen and Sørensen (2001) pointed out that the insured may receive reduced terminal expected utility by holding insurance products with guarantee, and it was clear the insurer's asset-liability management decisions must go beyond the monetary value of the insurance products. Døskeland and Nordahl (2008) reviewed the terminal expected utility of different insurance contracts under fair-pricing constraints; the evaluation result based on the expected utility theory was inconclusive and the cumulative prospect theory (CPT), one of the foremost behavioral economic models, was introduced as an alternative criteria. The aforementioned Jensen and Sørensen (2001); Consiglio et al. (2006); Døskeland and Nordahl (2008) did not explicitly consider the possibility of the bankruptcy and cessation of operation of the insurer in the process of calculating the terminal expected utility; on the conceptual basis laid by these articles, Chen and Hieber (2016) studied the early warning system (in Jørgensen (2007); Braun et al. (2011); Chen and Hieber (2016) it was called "traffic light system") and its contingency measures together with the relationship between asset allocation rules, default probabilities, the insured's terminal expected utility, and the additional effects of other contingency measures such as the capital injection.

In this work we complete the results in Chen and Hieber (2016); Chen et al. (2020) by analyzing an additional case which is simply the combination of previously covered ones. Meanwhile, we try to simplify the exposition and provide workable code in our GitHub repository. The remainder of the paper is organized as follows. Section 2 introduces the underlying asset model, the insurance contract, and the pertinent single-period expected utility framework setup. Section 3 introduces the probability apparatus, the contingent regulatory schemes, and the associated expected utility and default probability expressions. Section 4 presents the numerical results of the standard and extended expected utility maximization problem; Section 5 concludes.

2. The Model Setup

Following the setup originated in Briys and de Varenne (1994, 1997) and adopted by subsequent studies, e.g. Grosen and Jørgensen (2002); Chen and Hieber (2016); Hwang et al. (2015), we assume that, in a continous-time economy without tax effects, transaction costs and liquidity concerns, the policyholder and the equity holder agree to form an insurance company. Each of the two parties invests in the participating life insurance contract with maturity T years. Initially the entire value of investment is a_0; contribution of the policyholder (the initial premium) l_0 is $\alpha \cdot a_0$, where $0 < \alpha < 1$ is the wealth distribution coefficient. Under the physical measure $\mathbb{P}$ the investment a is distributed between the cash c with the price dynamics

$$\mathrm{d}c = r\, c\, \mathrm{d}t \tag{1}$$

where r is the constant risk-free interest rate, and the risky asset s with the price dynamics

$$\mathrm{d}s = s\,(\mu\, \mathrm{d}t + \sigma\, \mathrm{d}z) \tag{2}$$

where μ is the annual return, σ is the volatility, and z is the standard Wiener process. Set

$$a = w\, s + (1-w)c \tag{3}$$

where w is the constant weight invested in the risky asset; a is called the asset process with weight w. For the self-financing portfolio a, we have

$$\begin{aligned} \frac{\mathrm{d}a}{a} &= w\,\frac{\mathrm{d}s}{s} + (1-w)\,\frac{\mathrm{d}c}{c} \\ &= w\,(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}z) + (1-w)\,r\,\mathrm{d}t \\ &= (r + w\,(\mu - r))\,\mathrm{d}t + w\,\sigma\,\mathrm{d}z \end{aligned} \tag{4}$$

Under the (unique) risk-neutral measure $\mathbb{Q}$, (4) becomes

$$\frac{\mathrm{d}a}{a} = r\,\mathrm{d}t + w\,\sigma\,\mathrm{d}\widetilde{z} \tag{5}$$

with the transformed Wiener process $\widetilde{z} = z + \frac{\mu - r}{\sigma}\, t$. Note that in these descriptions we suppress the subscript t of stochastic processes a, c, and s for convenience.

The guaranteed payment l_t to the policyholder at time $t \in [0, T]$ is $l_t \equiv l_0 \cdot e^{\rho t}$, continuously accrued at guaranteed rate $\rho \leqslant r$. We stipulate that the insurance company defaults at time t if the asset value a_t drops below the default threshold $d_t \equiv d_0 \cdot e^{\rho t}$ with the initial default threshold value d_0; the default time τ is defined by

$$\tau = \inf\{t > 0 \,|\, a_t < d_t\} \tag{6}$$

For the setup of early warning system we introduce the regulatory boundary $k_t \equiv k_0 \cdot e^{\rho t}$ with initial regulatory threshold value $k_0 > d_0$; the regulatory boundary k_t lies above the default boundary d_t. Time for regulatory action $\hat{\tau}$ is defined by

$$\hat{\tau} = \inf\{t > 0 \,|\, a_t < k_t\} \tag{7}$$

Evidently $a_0 > k_0 > d_0$. With the existing default and regulatory boundaries in place, we have the following scenarios as shown in Figure 1:

- the underlying asset survived till maturity where no regulatory nor default boundaries are hit;
- the underlying asset process has hit the regulatory boundary, under certain regulatory scheme the default boundary is avoided;
- the underlying asset process has hit the regulatory boundary, and the default boundary is hit despite under certain regulatory scheme.

The focus of the study is to highlight the effect of regulatory schemes on the underlying asset process. Hereafter we use the notation that expressions with subscript $++$ stand for the case that both the regulatory and the default boundaries are hit; expressions with subscript $+-$ stand for the case that the regulatory boundary is hit but the default boundary is not; expressions with subscript $-$ stand for the case of not hitting the regulatory boundary; the expression with subscript $+$ stands for the case that the default boundary is hit before maturity. The superscript j is used to distinguish the state of the underlying variables in j-th regulatory scheme. For instance, $\mathscr{I}^2_{++}$ stands for the terminal expected utility of the asset process that both the regulatory and the default boundaries are hit under scheme 2.

The maturity payoff to the policyholder $\Psi_l(a^j_T)$ with regard to a^j_T, the final portfolio value under regulatory scheme j, is

$$\begin{aligned}\Psi_l(a^j_T) &= \begin{cases} a^j_T & \text{if } a^j_T \leqslant l_T \\ l_T & \text{if } l_T < a^j_T \leqslant \frac{l_T}{\alpha} \\ l_T + \delta\left(\alpha\, a^j_T - l_T\right) & \text{otherwise} \end{cases} \\ &= l_T + \delta\left(\alpha\, a^j_T - l_T\right)^+ - \left(l_T - a^j_T\right)^+ \end{aligned} \tag{8}$$

The participation coefficient $0 \leqslant \delta \leqslant 1$ is the fraction of the surplus to the policyholder. The rebate given at premature default time τ is

$$\Upsilon_l(\tau) = \min\left\{l_\tau, (1-\beta)\, a^j_\tau\right\} \tag{9}$$

where $0 \leqslant \beta \leqslant 1$ is the ratio of liquidation cost. Combining (8) and (9), the total payment at maturity is

$$\Lambda_l(a^j_T) = \mathbb{1}_{\{\tau > T\}}\, \Psi_l(a^j_T) + \mathbb{1}_{\{\tau \leqslant T\}}\, e^{r(T-\tau)}\, \Upsilon_l(\tau) \tag{10}$$

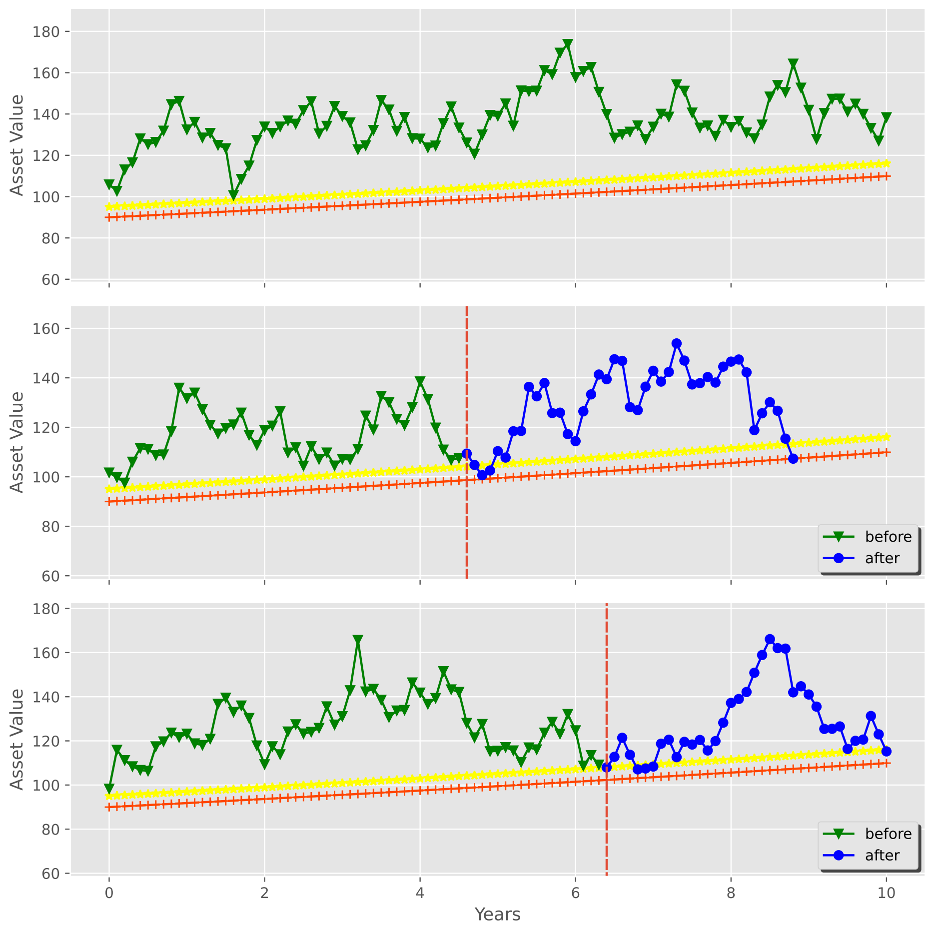

Figure 1: Scenarios of Asset Process.
Simulation of asset process by using the parameters taken from Table 1 where regulatory boundary (starred) and/or default boundary (crossed) are hit. Top: Regulatory boundary is not hit (case with subscript $-$). Bottom: Once the regulatory boundary is hit, capital injection and/or changing asset weight (circled) is applied and the asset is clear of default for the rest of its life (case with subscript $+-$). Middle: Both the regulatory boundary and the default boundary are hit (case with subscript $++$).

where $\mathbb{1}_{\{\cdot\}}$ stands for the indicator function which is 1 if the event in the curly brackets holds and 0 otherwise; the appearance of the factor $e^{r(T-\tau)}$ means that the rebate $\Upsilon_l(\tau)$ is collected only at maturity. From the perspective of the equityholder, the maturity payoff to the policyholder $\Psi_e(a_T^j)$ with regard to a_T^j is

$$\begin{aligned}\Psi_e(a_T^j) &= a_T^j - \Psi_l(a_T^j) \\ &= \begin{cases} 0 & \text{if } a_T^j \leqslant l_T \\ a_T^j - l_T & \text{if } l_T < a_T^j \leqslant \frac{l_T}{\alpha} \\ a_T^j - l_T - \delta\left(\alpha\, a_T^j - l_T\right) & \text{otherwise} \end{cases} \\ &= \left(a_T^j - l_T\right)^+ - \delta\left(\alpha\, a_T^j - l_T\right)^+ \end{aligned} \tag{11}$$

The rebate given at premature default time τ is

$$\Upsilon_e(\tau) = \max\left\{(1-\beta)\, a_\tau^j - l_\tau, 0\right\} \tag{12}$$

Combining (11) and (12), the total payment at maturity is

$$\Lambda_e(\tau) = \mathbb{1}_{\{\tau > T\}}\, \Psi_e(a_T^j) + \mathbb{1}_{\{\tau \leqslant T\}}\, e^{r(T-\tau)}\, \Upsilon_e(\tau) \tag{13}$$

Given the power utility function of the policyholder

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \tag{14}$$

with the risk aversion factor γ, the expectation of the utility of total payment at maturity

$$\mathscr{I}^j \equiv \mathsf{E}_{\mathbb{P}}\left\{u\big(\Lambda_l(a_T^j)\big)\right\} \tag{15}$$

or its certainty equivalent

$$\mathrm{ce}(\mathscr{I}^j) \equiv u^{-1}(\mathscr{I}^j) = \left((1-\gamma)\,\mathscr{I}^j\right)^{\frac{1}{1-\gamma}} \tag{16}$$

where u^{-1} is the inverse function/operation of u, is taken as the measure of the superiority of the regulatory scheme j; the best regulatory scheme maximizes the policyholder's terminal expected utility per initial premium under certain constraints to be specified later.

In Chen and Hieber (2016), Chen et al. (2020) three regulatory schemes in which action to be taken upon hitting the regulatory boundary are defined and tagged by $j = 0, 1, 2$ respectively, are proposed and investigated in detail:

- Scheme 0: Do nothing — Initialize the asset process with weight w_1 and the participation coefficient δ and keep them during the lifetime of the asset process.
- Scheme 1: Adjust the weight of the risky asset — Initialize the asset process with weight w_1 and the participation coefficient δ; adjust the weight to w_2 once the regulatory boundary is hit and keep it until the end.
- Scheme 2: Inject capital — Initialize the asset process with weight w and the participation coefficient δ; inject capital $\vartheta_{\hat{\tau}} = \nu\, k_{\hat{\tau}}$ where $0 \leqslant \nu \leqslant 1$ at $t = \hat{\tau}$.

Here we propose a natural extension of Scheme 2, which allows the change of the weight invested in the risky asset at the juncture of capital injection:

- Scheme 3: Inject capital and adjust the weight of the risky asset — Initialize the asset process with weight w_1 and the participation coefficient δ; inject capital $\vartheta_{\hat{\tau}} = \nu\, k_{\hat{\tau}}$ where $0 \leqslant \nu \leqslant 1$ and adjust the weight to w_2 at $t = \hat{\tau}$.

Denote the set of free variables of scheme j by Θ^j, we have

$$\Theta^0 \equiv (w, \delta), \qquad \Theta^1 \equiv (w_1, w_2, \delta), \qquad \Theta^2 \equiv (w, \nu, \delta), \qquad \Theta^3 \equiv (w_1, w_2, \nu, \delta). \tag{17}$$

The essential *fair contract condition* states that the initial premium should equal the expected discounted payoff; hence

$$l_0 = \mathsf{E}_{\mathbb{Q}} \left\{ e^{-rT} \Lambda_l(a_T^j) \right\} = \mathsf{E}_{\mathbb{Q}} \left\{ \mathbb{1}_{\{\tau > T\}}\, e^{-rT}\, \Psi_l(a_T^j) + \mathbb{1}_{\{\tau \leqslant T\}}\, e^{-r\tau}\, \Upsilon_l(\tau) \right\} \tag{18}$$

and equivalently

$$\begin{aligned} a_0 - l_0 = (1-\alpha)\, a_0 &= \mathsf{E}_{\mathbb{Q}} \left\{ e^{-rT} \Lambda_e(a_T^j) \right\} \\ &= \mathsf{E}_{\mathbb{Q}} \left\{ \mathbb{1}_{\{\tau > T\}}\, e^{-rT}\, \Psi_e(a_T^j) + \mathbb{1}_{\{\tau \leqslant T\}}\, e^{-r\tau}\, \Upsilon_e(\tau) \right\} \end{aligned} \tag{19}$$

Denote the expected discount payoff of the policyholder and the equityholder by

$$\mathscr{F}_l^j = \mathsf{E}_{\mathbb{Q}} \left\{ e^{-rT} \Lambda_l(a_T^j) \right\}, \qquad \mathscr{F}_e^j = \mathsf{E}_{\mathbb{Q}} \left\{ e^{-rT} \Lambda_e(a_T^j) \right\}, \tag{20}$$

the fair contract condition is written as

$$l_0 = \mathscr{F}_l^j \quad \text{or} \quad (1-\alpha)\, a_0 = \mathscr{F}_e^j. \tag{21}$$

The total initial premium is

$$\mathscr{L}^j = \begin{cases} l_0 & j = 0, 1 \\ l_0 + \vartheta_0 & j = 2, 3 \end{cases} \tag{22}$$

with the discounted injected capital ϑ_0

$$\vartheta_0 = \mathsf{E}_{\mathbb{Q}} \left\{ \mathbb{1}_{\{\hat{\tau} \leqslant T\}}\, e^{-r\hat{\tau}} \vartheta_{\hat{\tau}} \right\}. \tag{23}$$

It is clear that $\mathscr{I}^j$, P^j, $\mathscr{F}_l^j$ or $\mathscr{F}_e^j$, and $\mathscr{L}^j$ of each scheme j are all functions of its free variable set Θ^j; assume the fair contract condition (21) and perform the optimizations

$$\max_{\Theta^j} \frac{\mathrm{ce}(\mathscr{I}^j(\Theta^j))}{\mathscr{L}^j(\Theta^j)} \quad \text{s.t.} \quad \mathscr{F}_l^j(\Theta^j) = l_0 \quad \text{and} \quad \mathsf{P}^j(\Theta^j) \leqslant \epsilon, \qquad j = 0, 1, 2, 3. \tag{24}$$

where ϵ is the threshold probability $1-(1-\mathrm{PD})^T$ with the annualized default probability $\mathrm{PD} = 0.5\%$ which conforms to the solvency capital requirements (SCR) of Solvency II that the insurer should meet the obligations with 99.5% certainty within each fiscal year, the best overall regulatory scheme and the corresponding optimal investment strategy is selected by comparing the terminal expected utility per inital premium $\frac{\mathrm{ce}(\mathscr{I}^j)}{\mathscr{L}^j}$. Remarkably, as demonstrated in Chen and Hieber (2016), the optimization problem (24) with the *equality* constraint is equivalent to

$$\max_{\Theta^j} \frac{\mathrm{ce}(\mathscr{I}^j(\Theta^j))}{\mathscr{L}^j(\Theta^j)} \quad \text{s.t.} \quad \mathscr{F}_e^j(\Theta^j) \geqslant (1-\alpha)\, a_0 \quad \text{and} \quad \mathsf{P}^j(\Theta^j) \leqslant \epsilon, \qquad j = 0, 1, 2, 3. \tag{25}$$

with the *inequality* constraint.

It is instructive to work out the expected utility $\mathsf{E}_{\mathbb{P}}\left\{u(a_T)\right\}$ without payment at maturity and bankruptcy concerns (10), (15). Solution of the SDE (4) satisfied by the asset with weight w is

$$a_T = a_0\, e^{(r+w(\mu-r)-\frac{1}{2}w^2\sigma^2)\,T + w\sigma\sqrt{T}z}$$

where $z \sim N(0,1)$ is the standard normal distribution. By using the result (c.f. Buchen (2010, Theorem 3.1)) which states that

$$\mathsf{E}\big\{e^{\kappa z}F(z)\big\} = e^{\frac{1}{2}\kappa^2}\mathsf{E}\{F(z+\kappa)\}$$

for $z \sim N(0,1)$, $\kappa \in \mathbb{R}$ and $F(\cdot)$ a measurable function with finite expectation, $\mathsf{E}_{\mathbb{P}}\left\{u(a_T)\right\}$ is computed as

$$\begin{aligned} \mathsf{E}_{\mathbb{P}}\left\{u(a_T)\right\} &= \frac{a_0^{1-\gamma}}{1-\gamma}\, e^{(1-\gamma)(r+w(\mu-r)-\frac{1}{2}w^2\sigma^2)T}\, \mathsf{E}_{\mathbb{P}}\left\{e^{(1-\gamma)w\sigma\sqrt{T}z}\right\} \\ &= \frac{a_0^{1-\gamma}}{1-\gamma}\, e^{(1-\gamma)(r+w(\mu-r)-\frac{1}{2}w^2\sigma^2)T}\, e^{\frac{1}{2}(1-\gamma)^2 w^2\sigma^2 T} \\ &= \frac{a_0^{1-\gamma}}{1-\gamma}\, e^{(1-\gamma)(r+w(\mu-r)-\frac{1}{2}\gamma w^2\sigma^2)T} \end{aligned} \tag{26}$$

The maximum of $\mathsf{E}_{\mathbb{P}}\left\{u(a_T)\right\}$ is

$$\max\, \mathsf{E}_{\mathbb{P}}\left\{u(a_T)\right\} = \frac{a_0^{1-\gamma}}{1-\gamma}\, e^{(1-\gamma)\left(r+\frac{(\mu-r)^2}{2\gamma\sigma^2}\right)T} \tag{27}$$

which is attained at

$$w = \frac{\mu - r}{\gamma\sigma^2} \tag{28}$$

by inspecting the exponent of (26).

3. Theoretical Results

Set

$$\upmu(w) = r + w(\mu - r) - \rho - \frac{1}{2}w^2\sigma^2 \tag{29}$$

Define $f(t_0, t_1, p_0, p_1, w)$ as the probability of the asset process with weight w starting from initial position p_0 at time t_0 and hitting lower bound p_1 at time t_1, then

$$f(t_0, t_1, p_0, p_1, w) = \frac{\ln \frac{p_0}{p_1}}{\sqrt{2\pi}\, w\sigma\, (t_1 - t_0)^{\frac{3}{2}}} \exp\left(- \frac{\left(\ln \frac{p_0}{p_1} + \mu(w)(t_1 - t_0)\right)^2}{2\, w^2\sigma^2\, (t_1 - t_0)} \right) \tag{30}$$

Define $g(y, t_0, t_1, p_0, p_1, w)$ as the probability of the asset process with weight w starting from initial position p_0 at time t_0, not hitting lower bound p_1 and the final position is $\frac{1}{w\sigma} y$ at time t_1, then

$$\begin{aligned} & g(y, t_0, t_1, p_0, p_1, w) \\ = & \frac{1}{\sqrt{2\pi}\, w\sigma\, \sqrt{t_1 - t_0}} \exp\left(- \frac{\left(y - \mu(w)(t_1 - t_0)\right)^2}{2\, w^2\sigma^2\, (t_1 - t_0)} \right) \left\{ 1 - \exp\left(-2 \frac{\left(\ln \frac{p_0}{p_1}\right)^2 + y \ln \frac{p_0}{p_1}}{w^2\sigma^2 (t_1 - t_0)} \right) \right\} \end{aligned} \tag{31}$$

We delegate the proof of (30) and (31) to Appendix A. The above expressions f, g are derived under the physical measure $\mathbb{P}$; the corresponding $\mathbb{Q}$ measure version $\widetilde{f}$, $\widetilde{g}$ are formed by changing all the occurrences of $\mu(w)$ to $\widetilde{\mu}(w)$ with

$$\widetilde{\mu}(w) = r - \rho - \frac{1}{2} w^2\sigma^2. \tag{32}$$

Note that, the inverse Gaussian distribution $\mathrm{IG}(m, \lambda)$ has its probability distribution function (pdf)

$$\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(- \frac{\lambda (x - m)^2}{2 m^2 x} \right)$$

and (30) is the pdf of

$$\mathrm{IG}\left(\frac{\ln \frac{p_0}{p_1}}{\mu(w)}, \frac{\left(\ln \frac{p_0}{p_1}\right)^2}{w^2\sigma^2} \right)$$

With the probabilities (30), (31) and the convention in Figure 1, we explicitly write down the expressions $\mathscr{I}^j$, $\mathscr{F}_e^j$, and P^j of each regulatory scheme j needed for the formulation of the optimization problem (25).

Scheme 0: Do Nothing

The set of free variables Θ^0 is

$$\Theta^0 \equiv (w, \delta) \tag{33}$$

The terminal expected utility $\mathscr{I}^0$ is

$$\mathscr{I}^0(\Theta^0) = \mathscr{I}^0_+(\Theta^0) + \mathscr{I}^0_-(\Theta^0) \tag{34}$$

where

$$\mathscr{I}_{+}^{0}(\Theta^{0}) = \int_{0}^{T} u\big(e^{r(T-\tau)}\,\Upsilon_l(\tau)\big)\, f(0,\tau,a_0,d_0,w)\,\mathrm{d}\tau \tag{35}$$

$$\mathscr{I}_{-}^{0}(\Theta^{0}) = \int_{\ln\frac{d_0}{a_0}}^{\infty} u\big(\Psi_l\big(a_0\,e^{y+\rho T}\big)\big)\, g(y,0,T,a_0,d_0,w)\,\mathrm{d}y \tag{36}$$

The expected discount payoff of the equityholder $\mathscr{F}_e^0$ is

$$\mathscr{F}_e^{0}(\Theta^{0}) = \mathscr{F}_{+}^{0}(\Theta^{0}) + \mathscr{F}_{-}^{0}(\Theta^{0}) \tag{37}$$

where

$$\mathscr{F}_{+}^{0}(\Theta^{0}) = \int_{0}^{T} e^{-r\tau}\,\Upsilon_e(\tau)\,\widetilde{f}(0,\tau,a_0,d_0,w)\,\mathrm{d}\tau \tag{38}$$

$$\mathscr{F}_{-}^{0}(\Theta^{0}) = \int_{\ln\frac{d_0}{a_0}}^{\infty} e^{-rT}\,\Psi_e\big(a_0\,e^{y+\rho T}\big)\,\widetilde{g}(y,0,T,a_0,d_0,w)\,\mathrm{d}y \tag{39}$$

The default probability P^0 is

$$\mathsf{P}^{0}(\Theta^{0}) = \int_{0}^{T} f(0,\tau,a_0,d_0,w)\,\mathrm{d}\tau \tag{40}$$

Scheme 1: Adjust the Weight of the Risky Asset

The set of free variables Θ^1 is

$$\Theta^{1} \equiv (w_1, w_2, \delta) \tag{41}$$

The terminal expected utility $\mathscr{I}^1$ is

$$\mathscr{I}^{1}(\Theta^{1}) = \mathscr{I}_{++}^{1}(\Theta^{1}) + \mathscr{I}_{+-}^{1}(\Theta^{1}) + \mathscr{I}_{-}^{1}(\Theta^{1}) \tag{42}$$

where

$$\mathscr{I}_{++}^{1}(\Theta^{1}) = \int_{0}^{T}\int_{\hat\tau}^{T} u\big(e^{r(T-\tau)}\Upsilon_l(\tau)\big)\, f(0,\hat\tau,a_0,k_0,w_1)\, f(\hat\tau,\tau,k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \tag{43}$$

$$\mathscr{I}_{+-}^{1}(\Theta^{1}) = \int_{0}^{T}\int_{\ln\frac{d_0}{k_0}}^{\infty} u\big(\Psi_l\big(k_{\hat\tau}\,e^{y+\rho(T-\hat\tau)}\big)\big)\, f(0,\hat\tau,a_0,k_0,w_1)\, g(y,\hat\tau,T,k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}y\,\mathrm{d}\hat\tau \tag{44}$$

$$\mathscr{I}_{-}^{1}(\Theta^{1}) = \int_{\ln\frac{k_0}{a_0}}^{\infty} u\big(\Psi_l\big(a_0\,e^{y+\rho T}\big)\big)\, g(y,0,T,a_0,k_0,w_1)\,\mathrm{d}y \tag{45}$$

The expected discount payoff of the equityholder $\mathscr{F}_e^1$ is

$$\mathscr{F}_e^{1}(\Theta^{1}) = \mathscr{F}_{++}^{1}(\Theta^{1}) + \mathscr{F}_{+-}^{1}(\Theta^{1}) + \mathscr{F}_{-}^{1}(\Theta^{1}) \tag{46}$$

where

$$\mathscr{F}^1_{++}(\Theta^1) = \int_0^T \int_{\hat\tau}^T e^{-r\tau}\, \Upsilon_l(\tau)\, \widetilde{f}(0,\hat\tau,a_0,k_0,w_1)\, \widetilde{f}(\hat\tau,\tau,k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \tag{47}$$

$$\mathscr{F}^1_{+-}(\Theta^1) = \int_0^T \int_{\ln\frac{d_0}{k_0}}^\infty e^{-rT}\, \Psi_l\big(k_{\hat\tau}\, e^{y+\rho(T-\hat\tau)}\big)\, \widetilde{f}(0,\hat\tau,a_0,k_0,w_1)\, \widetilde{g}(y,\hat\tau,T,k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}y\,\mathrm{d}\hat\tau \tag{48}$$

$$\mathscr{F}^1_{-}(\Theta^1) = \int_{\ln\frac{k_0}{a_0}}^\infty e^{-rT}\, \Psi_l\big(a_0\, e^{y+\rho T}\big)\, \widetilde{g}(y,0,T,a_0,k_0,w_1)\,\mathrm{d}y \tag{49}$$

The default probability P^1 is

$$\mathsf{P}^1(\Theta^1) = \int_0^T \int_{\hat\tau}^T f(0,\hat\tau,a_0,k_0,w_1)\, f(\hat\tau,\tau,k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \tag{50}$$

Scheme 2: Inject Capital

The set of free variables Θ^2 is

$$\Theta^2 \equiv (w,\nu,\delta) \tag{51}$$

The terminal expected utility $\mathscr{I}^2$ is

$$\mathscr{I}^2(\Theta^2) = \mathscr{I}^2_{++}(\Theta^2) + \mathscr{I}^2_{+-}(\Theta^2) + \mathscr{I}^2_{-}(\Theta^2) \tag{52}$$

where

$$\mathscr{I}^2_{++}(\Theta^2) = \int_0^T \int_{\hat\tau}^T u\big(e^{r(T-\tau)}\Upsilon_l(\tau)\big)\, f(0,\hat\tau,a_0,k_0,w)\, f(\hat\tau,\tau,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \tag{53}$$

$$\mathscr{I}^2_{-}(\Theta^2) = \int_{\ln\frac{k_0}{a_0}}^\infty u\big(\Psi_l\big(a_0\, e^{y+\rho T}\big)\big)\, g(y,0,T,a_0,k_0,w)\,\mathrm{d}y \tag{54}$$

and

$$\mathscr{I}^2_{+-}(\Theta^2) = \int_0^T \int_{\ln\frac{d_{\hat\tau}}{\vartheta_{\hat\tau}+k_{\hat\tau}}}^\infty u\big(\Psi_l\big((\vartheta_{\hat\tau}+k_{\hat\tau})\, e^{y+\rho(T-\hat\tau)}\big)\big) \cdot f(0,\hat\tau,a_0,k_0,w)\cdot g(y,\hat\tau,T,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w)\,\mathrm{d}y\,\mathrm{d}\hat\tau \tag{55}$$

The expected discount payoff of the equityholder $\mathscr{F}^2_e$ is

$$\mathscr{F}^2_e(\Theta^2) = \mathscr{F}^2_{++}(\Theta^2) + \mathscr{F}^2_{+-}(\Theta^2) + \mathscr{F}^2_{-}(\Theta^2) \tag{56}$$

where

$$\mathscr{F}^2_{++}(\Theta^2) = \int_0^T \int_{\hat\tau}^T e^{-r\tau}\, \Upsilon_l(\tau)\, \widetilde{f}(0,\hat\tau,a_0,k_0,w)\, \widetilde{f}(\hat\tau,\tau,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \tag{57}$$

$$\mathscr{F}^2_{-}(\Theta^2) = \int_{\ln\frac{k_0}{a_0}}^\infty e^{-rT}\, \Psi_l\big(a_0\, e^{y+\rho T}\big)\, \widetilde{g}(y,0,T,a_0,k_0,w)\,\mathrm{d}y \tag{58}$$

and

$$\mathscr{F}^2_{+-}(\Theta^2) = \int_0^T \int_{\ln \frac{d_{\hat\tau}}{\vartheta_{\hat\tau}+k_{\hat\tau}}}^{\infty} e^{-rT}\, \Psi_l\big((\vartheta_{\hat\tau} + k_{\hat\tau})\, e^{y+\rho(T-\hat\tau)}\big) \cdot \widetilde{f}(0,\hat\tau,a_0,k_0,w) \cdot \widetilde{g}(y,\hat\tau,T,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w)\,\mathrm{d}y\,\mathrm{d}\hat\tau \quad (59)$$

The default probability P^2 is

$$\mathsf{P}^2(\Theta^2) = \int_0^T \int_{\hat\tau}^T f(0,\hat\tau,a_0,k_0,w)\, f(\hat\tau,\tau,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \quad (60)$$

The discounted injected capital ϑ_0 is

$$\vartheta_0(\Theta^2) = \mathsf{E}_{\mathbb{Q}}\left\{\mathbb{1}_{\{\hat\tau\leqslant T\}}\, e^{-r\hat\tau}\vartheta_{\hat\tau}\right\} = \int_0^T e^{-r\hat\tau}\vartheta_{\hat\tau}\, \widetilde{f}(0,\hat\tau,a_0,k_0,w)\,\mathrm{d}\hat\tau \quad (61)$$

Scheme 3: Inject Capital and Adjust the Weight of the Risky Asset

The set of free variables Θ^3 is

$$\Theta^3 \equiv (w_1, w_2, \nu, \delta) \quad (62)$$

The terminal expected utility $\mathscr{I}^3$ is

$$\mathscr{I}^3(\Theta^3) = \mathscr{I}^3_{++}(\Theta^3) + \mathscr{I}^3_{+-}(\Theta^3) + \mathscr{I}^3_{-}(\Theta^3) \quad (63)$$

where

$$\mathscr{I}^3_{++}(\Theta^3) = \int_0^T \int_{\hat\tau}^T u\big(e^{r(T-\tau)}\Upsilon_l(\tau)\big)\, f(0,\hat\tau,a_0,k_0,w_1)\, f(\hat\tau,\tau,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \quad (64)$$

$$\mathscr{I}^3_{-}(\Theta^3) = \int_{\ln\frac{k_0}{a_0}}^{\infty} u\big(\Psi_l\big(a_0\, e^{y+\rho T}\big)\big)\, g(y,0,T,a_0,k_0,w_1)\,\mathrm{d}y \quad (65)$$

and

$$\mathscr{I}^3_{+-}(\Theta^3) = \int_0^T \int_{\ln \frac{d_{\hat\tau}}{\vartheta_{\hat\tau}+k_{\hat\tau}}}^{\infty} u\big(\Psi_l\big((\vartheta_{\hat\tau} + k_{\hat\tau})\, e^{y+\rho(T-\hat\tau)}\big)\big) \cdot f(0,\hat\tau,a_0,k_0,w_1) \cdot g(y,\hat\tau,T,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}y\,\mathrm{d}\hat\tau \quad (66)$$

The expected discount payoff of the equityholder $\mathscr{F}^3_e$ is

$$\mathscr{F}^3_e(\Theta^3) = \mathscr{F}^3_{++}(\Theta^3) + \mathscr{F}^3_{+-}(\Theta^3) + \mathscr{F}^3_{-}(\Theta^3) \quad (67)$$

where

$$\mathscr{F}^3_{++}(\Theta^3) = \int_0^T \int_{\hat\tau}^T e^{-r\tau}\, \Upsilon_l(\tau)\, \widetilde{f}(0,\hat\tau,a_0,k_0,w_1)\, \widetilde{f}(\hat\tau,\tau,\vartheta_{\hat\tau}+k_{\hat\tau},d_{\hat\tau},w_2)\,\mathrm{d}\tau\,\mathrm{d}\hat\tau \quad (68)$$

$$\mathscr{F}^3_{-}(\Theta^3) = \int_{\ln\frac{k_0}{a_0}}^{\infty} e^{-rT}\, \Psi_l\big(a_0\, e^{y+\rho T}\big)\, \widetilde{g}(y,0,T,a_0,k_0,w_1)\,\mathrm{d}y \quad (69)$$

and

$$\mathscr{F}^3_{+-}(\Theta^3) = \int_0^T \int_{\ln \frac{d_{\hat{\tau}}}{\vartheta_{\hat{\tau}}+k_{\hat{\tau}}}}^{\infty} e^{-rT}\, \Psi_l\big((\vartheta_{\hat{\tau}} + k_{\hat{\tau}})\, e^{y+\rho(T-\hat{\tau})}\big) \\ \cdot \widetilde{f}(0, \hat{\tau}, a_0, k_0, w_1) \cdot \widetilde{g}(y, \hat{\tau}, T, \vartheta_{\hat{\tau}} + k_{\hat{\tau}}, d_{\hat{\tau}}, w_2)\, \mathrm{d}y\, \mathrm{d}\hat{\tau} \quad (70)$$

The default probability P^3 is

$$\mathsf{P}^3(\Theta^3) = \int_0^T \int_{\hat{\tau}}^T f(0, \hat{\tau}, a_0, k_0, w_1)\, f(\hat{\tau}, \tau, \vartheta_{\hat{\tau}} + k_{\hat{\tau}}, d_{\hat{\tau}}, w_1)\, \mathrm{d}\tau\, \mathrm{d}\hat{\tau} \quad (71)$$

The discounted injected capital ϑ_0 is

$$\vartheta_0(\Theta^3) = \mathsf{E}_{\mathbb{Q}}\left\{\mathbb{1}_{\{\hat{\tau}\leqslant T\}}\, e^{-r\hat{\tau}}\vartheta_{\hat{\tau}}\right\} = \int_0^T e^{-r\hat{\tau}}\vartheta_{\hat{\tau}}\, \widetilde{f}(0, \hat{\tau}, a_0, k_0, w_1)\, \mathrm{d}\hat{\tau} \quad (72)$$

4. Numerical Illustrations

We use `SciPy` commands `quad` and `nquad` for numerical integration and `minimize`, `shgo` for constrained optimization. The backbone of the numerical integration commands `quad`, `nquad` is the venerable `QUADPACK` (Piessens et al. (1983)) with adaptive evaluation and error estimation.

In Table 1 we list the model parameters needed for our numerical results, which is identical to the parameter set in Chen et al. (2020) for comparison purposes.

Parameter	Value	Definition
γ	3	risk aversion factor
σ	0.2	volatility of risky asset
μ	0.06	annual return of risky asset
r	0.025	risk-free interest rate
ρ	0.02	guaranteed rate
a_0	100	initial asset value
α	0.95	wealth distribution coefficient
l_0	95	initial premium $\equiv \alpha \cdot a_0$
k_0	95	initial regulatory threshold value
d_0	90, 94	initial default threshold value
β	0, 0.1	ratio of liquidation cost
T	10	maturity of the participating contract
ϵ	0.049	threshold probability $\equiv 1 - (1 - 0.5\%)^T$

Table 1: Model Parameters.

We first perform the standard terminal expected utility maximization under the fixed regulatory threshold value k_0 with combinations of the prescribed default threshold value

d_0 and the ratio of liquidation cost β as in Chen et al. (2020, Table 1); results are listed in Table 4. Table 2 lists the computed solutions in Chen et al. (2020). The indicators computed using the maximizers provided in Chen et al. (2020, Table 1) are listed in Table 3. Solutions are grouped by d_0 and β and ordered by the regulatory scheme j.

d_0	β	Θ	$\mathscr{L}$	ce	ce / $\mathscr{L}$	PD
90	0	[0.141, 0.83]	95	125.55	1.3216	0.005
90	0	[0.237, 0.068, 0.745]	95	126.03	1.3266	0.005
90	0	[0.286, 0.158, 0.975]	107.11	143.05	1.3355	0.005
90	0.1	[0.115, 0.867]	95	124.88	1.3145	0.0016
90	0.1	[0.231, 0.038, 0.727]	95	125.77	1.3239	0.0000
90	0.1	[0.241, 0.143, 0.975]	101.49	134.52	1.3254	0.0027
94	0	[0.096, 0.86]	95	124.56	1.3111	0.005
94	0	[0.181, 0.024, 0.839]	95	125.23	1.3183	0.0001
94	0	[0.267, 0.186, 1.0]	113.61	151.19	1.3308	0.005
94	0.1	[0.072, 0.937]	95	124.19	1.3072	0.0009
94	0.1	[0.179, 0.02, 0.844]	95	125.23	1.3182	0.0000
94	0.1	[0.247, 0.173, 1.0]	106.11	140.04	1.3198	0.0042

Table 2: Results of the Standard Utility Maximization Problem Reported in Chen et al. (2020).

d_0	β	Θ	$\mathscr{L}$	ce	ce / $\mathscr{L}$	PD
90	0	[0.141, 0.83]	95.000000	125.546161	1.321539	0.004967
90	0	[0.237, 0.068, 0.745]	95.000000	125.011988	1.315916	0.000455
90	0	[0.286, 0.158, 0.975]	105.913652	141.313859	1.334236	0.005027
90	0.1	[0.115, 0.867]	95.000000	124.879234	1.314518	0.001642
90	0.1	[0.231, 0.038, 0.727]	95.000000	124.383957	1.309305	0.000000
90	0.1	[0.241, 0.143, 0.975]	104.021604	137.582285	1.322632	0.002697
94	0	[0.096, 0.86]	95.000000	124.573330	1.311298	0.005052
94	0	[0.181, 0.024, 0.839]	95.000000	125.240784	1.318324	0.000172
94	0	[0.267, 0.186, 1.0]	107.424510	142.959960	1.330795	0.005013
94	0.1	[0.072, 0.937]	95.000000	124.185083	1.307211	0.000869
94	0.1	[0.179, 0.02, 0.844]	95.000000	125.231098	1.318222	0.000019
94	0.1	[0.247, 0.173, 1.0]	106.074504	139.998613	1.319814	0.004224

Table 3: Listings of Indicators Computed by Using the Maximizers of Chen et al. (2020).

The first step towards the expected utility maximization problem is the accurate evaluation of the integral expressions $\mathscr{I}^j$ and P^j under the parameter sets Θ^j. Comparing Table 2 with Table 3, one can see that both the ce and PD values are in good agreement under regulatory scheme 0. The remaining PD values under regulatory schemes 1, 2

d_0	β	Θ	$\mathscr{L}$	ce	ce$/\mathscr{L}$	PD
90	0	[0.141204 0.830309]	95.000000	125.554902	1.321631	0.005000
90	0	[0.226730 0.108312 0.787944]	95.000000	125.923260	1.325508	0.005000
90	0	[0.294667 0.168961 0.993831]	106.829027	142.881186	1.337475	0.005000
90	0	[0.462946 0.277238 0.174766 1.]	109.141419	146.857189	1.345568	0.005000
90	0.1	[0.115098 0.866459]	95.000000	124.875335	1.314477	0.001651
90	0.1	[0.204847 0.072238 0.804288]	95.000000	125.487768	1.320924	0.000588
90	0.1	[0.258167 0.159739 0.999517]	105.482527	139.940490	1.326670	0.002984
90	0.1	[0.379633 0.194787 0.127692 1.]	104.808861	140.134804	1.337051	0.001983
94	0	[0.095793 0.859658]	95.000000	124.562267	1.311182	0.005000
94	0	[0.183595 0.024731 0.836624]	95.000000	125.234064	1.318253	0.000235
94	0	[0.266554 0.185641 1.000000]	107.389862	142.911819	1.330776	0.005000
94	0	[0.419212 0.219647 0.161264 1.]	107.737506	144.592122	1.342078	0.005000
94	0.1	[0.072022 0.936933]	95.000000	124.185048	1.307211	0.000871
94	0.1	[0.179419 0.019929 0.843590]	95.000000	125.227378	1.318183	0.000018
94	0.1	[0.245744 0.172570 1.000000]	106.014155	139.915387	1.319780	0.004151
94	0.1	[0.405692 0.189453 0.160658 1.]	107.578890	143.259427	1.331669	0.002592

Table 4: Results of the Standard Utility Maximization Problem.

are fairly close (results in 3 are shown in 4 decimal places at most, however), expect for the case $d_0 = 90, \beta = 0$ under scheme 1. Most ce values in Table 2 are higher than their counterparts in Table 3, which means that the integrals involved are systematically undercalculated in Chen et al. (2020).

Our computational results in Table 4 reconfirm the main findings in Chen and Hieber (2016); Chen et al. (2020) which states that, given a typical set of model parameters with designated initial default and regulatory threshold values, the regulatory scheme 2 which injects capital once the asset level hits the early warning boundary provides the most terminal expected utility per initial premium. The observation made in their papers that the introduction of the liquidation cost ($\beta \neq 0$) inhibits both the risky asset investment (lower w) and the annualized default probability PD is still intact.

An immediate extension of the standard utility maximization problem is to treat k_0, d_0 as free variables with the natural constraint $k_0 > d_0$, i.e. to determine the optimal regulatory scheme under minimal constraints. Solutions of this extended maximization problem provide global upper bounds and give insights into the selection of regulatory schemes. We use the `shgo` command which implements the simplicial homology global optimization algorithm to search for the maximizer. The results are listed in Table 5.

For the extended utility maximization problem, regulatory scheme 2 still provides the most terminal expected utility per initial premium, albeit with very low defined bankruptcy level $d_0^\star$ and rather high injected capital ratio ν. Both the regulatory schemes 0 and 1 generate the same amount of the terminal expected utility per initial premium, and the initial weight of the risky asset is almost identical. The annualized default probability PD is negligible in all cases.

	Θ	$k_0^\star$	$d_0^\star$	$\mathscr{L}$	ce $/\mathscr{L}$
0	[0.348452 0.847450]	—	42.745749	95.000000	1.350903
1	[0.352068 0.339928 0.845708]	86.997866	6.680755	95.000000	1.350929
2	[0.353410 0.030193 0.848576]	66.763865	2.940564	95.111791	1.350979
3	[0.349708 0.291281 0.055949 0.847715]	56.038952	43.586759	95.019544	1.350910

Table 5: Results of the Extended Utility Maximization Problem.

5. Conclusion

We assess four contingent regulatory schemes under the early warning monitoring system by maximizing the terminal expected utility. The explicit consideration of the default event for the underlying asset portfolio which follows the probability law of geometric Brownian motion leads to integral expressions comprised of the density function of certain time-inhomogeneous inverse Gaussian process; adaptive numerical integration is employed throughout the maximization process. The numerical result shows that, among all contingent regulatory measures, the capital injection with weight adjustment of risky asset provides the insurer who is under financial distress the greatest terminal expected utility per initial premium.

In the present work we only consider the asset portfolio which follows the probabilty law of the geometric Brownian motion. A perusal of the expressions in section 3 reveals that integrals of terminal expected utility and default probability hinge on the model-dependent probabilities f, g and the $\mathbb{Q}$ measure version $\widetilde{f}$, $\widetilde{g}$, so the results obtained may readily be extended to other portfolio models once the aforementioned probabilities are determined; the pursuit of this avenue of research is left for future work.

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Appendix A. Proof of (30) and (31)

Let z be a Brownian motion without drift and define the running maximum process $z_t^\star = \sup_{0\leqslant s\leqslant t} z_s$ and the running minimum process $z_t^\circ = \inf_{0\leqslant s\leqslant t} z_s$. First we quote the following standard (c.f. Jeanblanc et al. (2009, Proposition 3.1.1.1), Baldi (2017, Theorem 3.4))

Result. For $\alpha \geqslant 0$ and $\alpha \geqslant \beta$

$$\mathsf{P}(z_t^\star \geqslant \alpha, z_t \leqslant \beta) = \mathsf{P}(z_t \geqslant 2\alpha - \beta) \tag{A.1}$$

The joint probability density function $\varphi_{z_t^\star, z_t}$ is thus

$$\begin{aligned}\varphi_{z_t^\star, z_t}(\alpha,\beta) &= \frac{\partial^2}{\partial\alpha\partial\beta}\mathsf{P}(z_t^\star \leqslant \alpha, z_t \leqslant \beta) = \frac{\partial^2}{\partial\alpha\partial\beta}\big(\mathsf{P}(z_t \leqslant \beta) - \mathsf{P}(z_t^\star \geqslant \alpha, z_t \leqslant \beta)\big)\\ &= -\frac{\partial^2}{\partial\alpha\partial\beta}\mathsf{P}(z_t^\star \geqslant \alpha, z_t \leqslant \beta) = -\frac{\partial^2}{\partial\alpha\partial\beta}\mathsf{P}(z_t \geqslant 2\alpha - \beta)\end{aligned}$$

Note that

$$\mathsf{P}(z_t \geqslant 2\alpha - \beta) = \frac{1}{\sqrt{2\pi t}}\int_{2\alpha-\beta}^{\infty} e^{-\frac{1}{2t}y^2}\mathrm{d}y$$

Performing the differentiation we have

$$\varphi_{z_t^\star, z_t}(\alpha,\beta) = \mathbb{1}_{\{\alpha \geqslant \max(\beta,0)\}}\,\frac{2\alpha-\beta}{t}\sqrt{\frac{2}{\pi t}}\,e^{-\frac{(2\alpha-\beta)^2}{2t}}$$

Application of the Girsanov theorem yields that the Brownian motion with constant drift $\widetilde{z}_t \equiv z_t + \mu\, t$ is a Brownian motion without drift under the transformed measure $\widetilde{\mathsf{P}}$ with the Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} = e^{-\mu z_t - \frac{1}{2}\mu^2 t}$$

where P is the original measure, then

$$\begin{aligned}\mathsf{P}(\widetilde{z}_t^\star \geqslant \alpha, \widetilde{z}_t \leqslant \beta) &= \mathsf{E}\left\{\mathbb{1}_{\{\widetilde{z}_t^\star \geqslant \alpha, \widetilde{z}_t \leqslant \beta\}}\right\} = \widetilde{\mathsf{E}}\left\{\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\widetilde{\mathsf{P}}}\,\mathbb{1}_{\{\widetilde{z}_t^\star \geqslant \alpha, \widetilde{z}_t \leqslant \beta\}}\right\}\\ &= \widetilde{\mathsf{E}}\left\{e^{\mu z_t + \frac{1}{2}\mu^2 t}\,\mathbb{1}_{\{\widetilde{z}_t^\star \geqslant \alpha, \widetilde{z}_t \leqslant \beta\}}\right\}\\ &= \widetilde{\mathsf{E}}\left\{e^{\mu \widetilde{z}_t - \frac{1}{2}\mu^2 t}\,\mathbb{1}_{\{\widetilde{z}_t^\star \geqslant \alpha, \widetilde{z}_t \leqslant \beta\}}\right\}\end{aligned}$$

Hence the joint probability density function $\varphi_{\widetilde{z}_t^\star,\widetilde{z}_t}$ is

$$\varphi_{\widetilde{z}_t^\star,\widetilde{z}_t}(\alpha,\beta) = \mathbb{1}_{\{\alpha\geqslant\max(\beta,0)\}}\,\frac{2\alpha-\beta}{t}\sqrt{\frac{2}{\pi t}}\;e^{\mu\beta-\frac{1}{2}\mu^2 t-\frac{(2\alpha-\beta)^2}{2t}} \tag{A.2}$$

Define

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,\mathrm{d}y, \qquad \phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Using (A.2), $\mathsf{P}(\widetilde{z}_t^\star \leqslant \alpha)$ is computed as

$$\begin{aligned}
\mathsf{P}(\widetilde{z}_t^\star \leqslant \alpha) &= \frac{1}{t}\sqrt{\frac{2}{\pi t}}\int_0^\alpha\int_{-\infty}^{x}(2x-y)\,e^{\mu y-\frac{1}{2}\mu^2 t-\frac{(2x-y)^2}{2t}}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{t}\sqrt{\frac{2}{\pi t}}e^{-\frac{1}{2}\mu^2 t}\Big(\int_0^\alpha\int_y^\alpha + \int_{-\infty}^0\int_0^\alpha\Big)(2x-y)\,e^{\mu y-\frac{(2x-y)^2}{2t}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{t}\sqrt{\frac{2}{\pi t}}e^{-\frac{1}{2}\mu^2 t}\frac{t}{2}\int_{-\infty}^{\alpha} e^{\mu y}\big(e^{-\frac{1}{2t}y^2} - e^{-\frac{(2a-y)^2}{2t}}\big)\,\mathrm{d}y\\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\alpha} e^{-\frac{(y-\mu t)^2}{2t}}\,\mathrm{d}y - e^{2\alpha\mu}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\alpha} e^{-\frac{(y-(2\alpha+\mu t))^2}{2t}}\,\mathrm{d}y\\
&= \Phi\Big(\frac{\alpha-\mu t}{\sqrt{t}}\Big) - e^{2\alpha\mu}\,\Phi\Big(\frac{-\alpha-\mu t}{\sqrt{t}}\Big)
\end{aligned} \tag{A.3}$$

By definition

$$\begin{aligned}
\mathsf{P}(\widetilde{z}_t^\circ \leqslant \alpha) &= \mathsf{P}\big(\inf_{0\leqslant s\leqslant t}(z_s+\mu s) \leqslant \alpha\big) = \mathsf{P}\big(-\sup_{0\leqslant s\leqslant t}(-z_s-\mu s) \leqslant \alpha\big)\\
&= \mathsf{P}\big(\sup_{0\leqslant s\leqslant t}(-z_s-\mu s) \geqslant -\alpha\big) = \mathsf{P}\big(\sup_{0\leqslant s\leqslant t}(z_s-\mu s) \geqslant -\alpha\big)\\
&= 1-\mathsf{P}\big(\sup_{0\leqslant s\leqslant t}(z_s-\mu s) \leqslant -\alpha\big)\\
&= 1-\Phi\Big(\frac{-\alpha+\mu t}{\sqrt{t}}\Big) + e^{2\alpha\mu}\,\Phi\Big(\frac{\alpha+\mu t}{\sqrt{t}}\Big)
\end{aligned} \tag{A.4}$$

Differentiate (A.4) with respect to t yields the distribution function

$$-\phi\Big(\frac{-\alpha+\mu t}{\sqrt{t}}\Big)\Big(\frac{\mu}{2\sqrt{t}}+\frac{\alpha}{2\,t^{\frac{3}{2}}}\Big) + e^{2\alpha\mu}\,\phi\Big(\frac{\alpha+\mu t}{\sqrt{t}}\Big)\Big(\frac{\mu}{2\sqrt{t}}-\frac{\alpha}{2\,t^{\frac{3}{2}}}\Big) = -\frac{\alpha}{\sqrt{2\pi t^3}}e^{-\frac{(\alpha-\mu t)^2}{2t}} \tag{A.5}$$

By applying the transformation $(\alpha,\beta,\mu)\mapsto(-\alpha,-\beta,-\mu)$ in (A.2), we obtain

$$\varphi_{\widetilde{z}_t^\circ,\widetilde{z}_t}(\alpha,\beta) = \mathbb{1}_{\{\alpha\leqslant\min(\beta,0)\}}\,\frac{\beta-2\alpha}{t}\sqrt{\frac{2}{\pi t}}\;e^{\mu\beta-\frac{1}{2}\mu^2 t-\frac{(2\alpha-\beta)^2}{2t}}$$

Hence

$$\int_\alpha^y \frac{y-2x}{t}\sqrt{\frac{2}{\pi t}}\;e^{\mu y-\frac{1}{2}\mu^2 t-\frac{(2x-y)^2}{2t}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-\mu t)^2}{2t}}\big(1-e^{\frac{2(\alpha y-\alpha^2)}{t}}\big) \tag{A.6}$$

Note that the boundary condition yields

$$p_0\, e^{(r+w(\mu-r)-\frac{1}{2}w^2\sigma^2)\,t+w\sigma z_t} \geqslant p_1\, e^{\rho t}$$

and is equivalent to

$$z_t + \frac{1}{w\sigma}\underbrace{\left(r + w(\mu - r) - \rho - \frac{1}{2}w^2\sigma^2\right)}_{\equiv \upmu(w)}\, t \geqslant \frac{1}{w\sigma}\ln\frac{p_1}{p_0}$$

So in (A.5), (A.6), let

$$\alpha = \frac{1}{w\sigma}\ln\frac{p_1}{p_0}, \qquad \mu = \frac{1}{w\sigma}\upmu(w), \qquad y = \frac{1}{w\sigma}y \tag{A.7}$$

and note that the Jacobian is $\frac{1}{w\sigma}$, (30) and (31) are recovered.