Inverse Obstacle Scattering of a Perfect Conductor

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0.1 Prerequisites

Proposition 0.1 (Green Formula)**.**

$$\int_\Omega (E \cdot \Delta H - H \cdot \Delta E)\, \mathrm{d}V = \int_\Gamma (E \times \operatorname{curl} H + E \operatorname{div} H - H \times \operatorname{curl} E - H \operatorname{div} E) \cdot \nu \,\mathrm{d}\sigma$$

If $\operatorname{div} E = \operatorname{div} H = 0$, then

$$\int_\Omega E \cdot \operatorname{curl}\operatorname{curl} H - H \cdot \operatorname{curl}\operatorname{curl} E \,\mathrm{d}V \tag{0.1}$$

$$= \int_\Gamma (E \times \operatorname{curl} H - H \times \operatorname{curl} E) \cdot \nu \,\mathrm{d}\sigma \tag{0.2}$$

$$= \int_\Gamma (\nu \times E) \cdot \operatorname{curl} H - (\nu \times H) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \tag{0.3}$$

Proposition 0.2 (Stratton-Chu Representation Formula)**.** If $E, H \in C^1(\Omega_+) \cap C(\Omega_+ \cup \Gamma)$ satisfy Maxwell equations in Ω_+ and the Silver-Müller radiation condition, then for $x \in \Omega_+$

$$E(x) = \operatorname{curl} \int_\Gamma \nu(x) \times E(y)\Phi(x,y) \,\mathrm{d}\sigma(y) + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\Phi(x,y) \,\mathrm{d}\sigma(y)$$

$$H(x) = \operatorname{curl} \int_\Gamma \nu(x) \times H(y)\Phi(x,y) \,\mathrm{d}\sigma(y) - \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x,y) \,\mathrm{d}\sigma(y).$$

Proposition 0.3 (Far Field Patterns)**.**

$$E^\infty(\hat{x}) = ik\, \hat{x} \times \int_\Gamma \{\nu(y) \times E(y) + (\nu(y) \times H(y)) \times \hat{x}\}\, e^{-ik\hat{x}\cdot y} \,\mathrm{d}\sigma(y)$$
$$H^\infty(\hat{x}) = ik\, \hat{x} \times \int_\Gamma \{\nu(y) \times H(y) - (\nu(y) \times E(y)) \times \hat{x}\}\, e^{-ik\hat{x}\cdot y} \,\mathrm{d}\sigma(y)$$

Proposition 0.4 (Rellich Lemma)**.** If $E, H \in C^1(\Omega_+)$ is a radiating solution of Maxwell equations such that the electric far field pattern vanishes identically, then $E = H = 0$ in Ω_+.

0.2 Reciprocity Relations

Assume $x, z \in \Omega_+$, $\hat{x}, d \in \mathbb{S}^2$, $p, q \in \mathbb{R}^3$.

Given the incident electromagentic wave

$$E^{\mathrm{i}}_{\mathrm{w}}(x,d,p) = \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d},$$
$$H^{\mathrm{i}}_{\mathrm{w}}(x,d,p) = \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) e^{ikx\cdot d},$$

the scattered field is denoted by

$$E^{\mathrm{s}}_{\mathrm{w}}(x,d,p), \quad H^{\mathrm{s}}_{\mathrm{w}}(x,d,p)$$

with corresponding far field pattern

$$E^{\infty}_{\mathrm{w}}(\hat{x},d,p), \quad H^{\infty}_{\mathrm{w}}(\hat{x},d,p).$$

Given the incident dipole

$$\begin{aligned} E^{\mathrm{i}}_{\mathrm{p}}(x,z,p) &= \frac{i}{k}\,\mathrm{curl}_x\,\mathrm{curl}_x\, p\Phi(x,z), \\ H^{\mathrm{i}}_{\mathrm{p}}(x,z,p) &= \mathrm{curl}_x\, p\Phi(x,z), \end{aligned}$$

the scattered field is denoted by

$$E^{\mathrm{s}}_{\mathrm{p}}(x,z,p), \quad H^{\mathrm{s}}_{\mathrm{p}}(x,z,p)$$

with the corresponding far field pattern

$$E^{\infty}_{\mathrm{p}}(\hat{x},z,p), \quad H^{\infty}_{\mathrm{p}}(\hat{x},z,p).$$

The total field is denoted by

$$\begin{aligned} E_{\mathrm{w}}(x,d,p) &= E^{\mathrm{i}}_{\mathrm{w}}(x,d,p) + E^{\mathrm{s}}_{\mathrm{w}}(x,d,p) \\ H_{\mathrm{w}}(x,d,p) &= H^{\mathrm{i}}_{\mathrm{w}}(x,d,p) + H^{\mathrm{s}}_{\mathrm{w}}(x,d,p) \\ E_{\mathrm{p}}(x,z,p) &= E^{\mathrm{i}}_{\mathrm{p}}(x,z,p) + E^{\mathrm{s}}_{\mathrm{p}}(x,z,p) \\ H_{\mathrm{p}}(x,z,p) &= H^{\mathrm{i}}_{\mathrm{p}}(x,z,p) + H^{\mathrm{s}}_{\mathrm{p}}(x,z,p) \end{aligned}$$

Theorem 0.1 (Mixed Reciprocity Relation)**.**

$$p \cdot E^{\mathrm{s}}_{\mathrm{w}}(z,-\hat{x},q) = 4\pi q \cdot E^{\infty}_{\mathrm{p}}(\hat{x},z,p)$$

Proof. From proposition (0.3) we have

$$\begin{aligned} 4\pi q \cdot E^{\infty}_{\mathrm{p}}(\hat{x},z,p) = \int_\Gamma & \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y,z,p) \cdot H^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q) \\ & + \nu(y) \times H^{\mathrm{s}}_{\mathrm{p}}(y,z,p) \cdot E^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) \end{aligned} \tag{0.4}$$

From Green formula (0.1) we have

$$\begin{aligned} \int_\Gamma & \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y,z,p) \cdot H^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q) \\ & + \nu(y) \times H^{\mathrm{s}}_{\mathrm{p}}(y,z,p) \cdot E^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) = 0 \end{aligned} \tag{0.5}$$

Add (0.4), (0.5) and apply the boundary condition

$$\nu(y) \times E_{\mathrm{w}}(y,-\hat{x},q) = 0 \quad \forall y \in \Gamma$$

we have

$$4\pi q \cdot E^{\infty}_{\mathrm{p}}(\hat{x},z,p) = \int_\Gamma \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y,z,p) \cdot H_{\mathrm{w}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) \tag{0.6}$$

From Stratton-Chu representation,

$$E_{\rm w}^{\rm s}(z,-\hat{x},q) = \operatorname{curl}\int_\Gamma \nu(y)\times E_{\rm w}^{\rm s}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H_{\rm w}^{\rm s}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y) \quad (0.7)$$

From Green formula (0.1),

$$0 = \operatorname{curl}\int_\Gamma \nu(y)\times E_{\rm w}^{\rm i}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H_{\rm w}^{\rm i}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y) \quad (0.8)$$

Add (0.7), (0.8) and apply the boundary condition

$$\nu(y)\times E_{\rm w}(y,-\hat{x},q) = 0 \quad \forall y\in\Gamma$$

we have

$$E_{\rm w}^{\rm s}(z,-\hat{x},q) = \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H_{\rm w}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y) \quad (0.9)$$

From (0.9), the identity

$$p\cdot\operatorname{curl}\operatorname{curl}_z\{a(y)\Phi(z,y)\} = a(y)\cdot\operatorname{curl}\operatorname{curl}_z\{p\Phi(z,y)\},$$

and the boundary condition

$$\nu(y)\times E_{\rm p}^{\rm i}(y,z,p) = -\nu(y)\times E_{\rm p}^{\rm s}(y,z,p) \quad \forall y\in\Gamma$$

we have

$$\begin{aligned} p\cdot E_{\rm w}^{\rm s}(z,-\hat{x},q) &= \frac{i}{k}\,p\cdot\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H_{\rm w}(y,-\hat{x},q)\Phi(z,y)\,{\rm d}\sigma(y)\\ &= \frac{i}{k}\int_\Gamma \nu(y)\times H_{\rm w}(y,-\hat{x},q)\cdot\operatorname{curl}\operatorname{curl}\{p\Phi(z,y)\}\,{\rm d}\sigma(y)\\ &= \int_\Gamma \nu(y)\times H_{\rm w}(y,-\hat{x},q)\cdot E_{\rm p}^{\rm i}(y,z,p)\,{\rm d}\sigma(y)\\ &= -\int_\Gamma \nu(y)\times E_{\rm p}^{\rm i}(y,z,p)\cdot H_{\rm w}(y,-\hat{x},q)\,{\rm d}\sigma(y)\\ &= \int_\Gamma \nu(y)\times E_{\rm p}^{\rm s}(y,z,p)\cdot H_{\rm w}(y,-\hat{x},q)\,{\rm d}\sigma(y), \end{aligned}$$

which equals (0.6). □

Theorem 0.2 (Reciprocity Relation)**.**

$$q\cdot E_{\rm w}^{\infty}(\hat{x},d,p) = p\cdot E_{\rm w}^{\infty}(-d,-\hat{x},q)$$

Proof. Apply Green formula (0.1) to $E^{\mathrm{i}}_{\mathrm{w}}$ in Ω_-, $E^{\mathrm{s}}_{\mathrm{w}}$ in Ω_+, we have

$$\int_\Gamma \nu(y) \times E^{\mathrm{i}}_{\mathrm{w}}(y,d,p) \cdot H^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q) - \nu(y) \times E^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q) \cdot H^{\mathrm{i}}_{\mathrm{w}}(y,d,p)\,\mathrm{d}\sigma(y) = 0 \quad (0.10)$$

$$\int_\Gamma \nu(y) \times E^{\mathrm{s}}_{\mathrm{w}}(y,d,p) \cdot H^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q) - \nu(y) \times E^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q) \cdot H^{\mathrm{s}}_{\mathrm{w}}(y,d,p)\,\mathrm{d}\sigma(y) = 0 \quad (0.11)$$

From proposition (0.3) we have

$$4\pi q \cdot E^{\infty}_{\mathrm{w}}(\hat{x},d,p) = \int_\Gamma \nu(y) \times E^{\mathrm{s}}_{\mathrm{w}}(y,d,p) \cdot H^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q) + \nu(y) \times H^{\mathrm{s}}_{\mathrm{w}}(y,d,p) \cdot E^{\mathrm{i}}_{\mathrm{w}}(y,-\hat{x},q)\,\mathrm{d}\sigma(y) \quad (0.12)$$

Interchange p, q and $d, \hat{x}$ respectively in (0.12), we have

$$4\pi q \cdot E^{\infty}_{\mathrm{w}}(\hat{x},d,p) = \int_\Gamma \nu(y) \times E^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q) \cdot H^{\mathrm{i}}_{\mathrm{w}}(y,d,p) + \nu(y) \times H^{\mathrm{s}}_{\mathrm{w}}(y,-\hat{x},q) \cdot E^{\mathrm{i}}_{\mathrm{w}}(y,d,p)\,\mathrm{d}\sigma(y) \quad (0.13)$$

Subtract (0.12) with (0.13) and add (0.10), (0.11), together with the boundary condition

$$\nu(y) \times E_{\mathrm{w}}(y,d,p) = \nu(y) \times E_{\mathrm{w}}(y,-\hat{x},p) = 0 \quad \forall y \in \Gamma$$

the result follows. □

0.3 A Uniqueness Theorem

Theorem 0.3. If D_1 and D_2 are two perfect conductors such that the electric far field patterns coincide for a fixed wave number, all incident directions and all observation directions, then $D_1 = D_2$.

Proof. Let U be the unbounded component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. By Rellich lemma,

$$E^{\mathrm{s}}_{\mathrm{w},1}(x,d,p) = E^{\mathrm{s}}_{\mathrm{w},2}(x,d,p) \quad \forall x \in U, d, p \in \mathbb{S}^2.$$

By mixed reciprocity relation,

$$E^{\infty}_{\mathrm{w},1}(\hat{x},z,p) = E^{\infty}_{\mathrm{w},2}(\hat{x},z,p) \quad \forall z \in U, \hat{x}, p \in \mathbb{S}^2.$$

Again by Rellich lemma,

$$E^{\mathrm{s}}_{\mathrm{p},1}(x,z,p) = E^{\mathrm{s}}_{\mathrm{p},2}(x,z,p) \quad \forall x, z \in U, p \in \mathbb{S}^2.$$

Assume $D_1 \neq D_2$, then $\exists\, \tilde{x} \in U$ such that $\tilde{x} \in \partial D_1, \tilde{x} \notin \overline{D_2}$. Construct $\{z_n\}$ such that $z_n = \tilde{x} + \frac{1}{n}\nu(\tilde{x}) \in U$ for sufficiently large n. From the well-posedness of the solution on D_2, $E^{\mathrm{s}}_{\mathrm{p},2}(\tilde{x},\tilde{x},p)$ is well-behaved. But

$$E^{\mathrm{s}}_{\mathrm{p},1}(\tilde{x},z_n,q) \to \infty \text{ as } z_n \to \tilde{x} \text{ and given } p \perp \nu(\tilde{x})$$

in order to fulfill the boundary condition with the incident dipole $E^{\mathrm{i}}_{\mathrm{p},1}(\tilde{x},z_n,p)$, which becomes unbounded as $z_n \to \tilde{x}$.

□

0.4 Factorization of the Far Field Operator

Here we set the function spaces which will be of use later.

1. $L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} = \{v \,|\, v \in L_2(\Gamma)^3,\ \nu \cdot v = 0,\ \mathrm{div}_\Gamma\, v \in L_2(\Gamma)\}$.
2. $L_{2,\mathrm{t}}^{\mathrm{curl}_\Gamma} = \{v \,|\, v \in L_2(\Gamma)^3,\ \nu \cdot v = 0,\ \mathrm{curl}_\Gamma\, v \in L_2(\Gamma)\}$.

Proposition 0.5. $v \to \nu \times v$ is an isomorphism from $L_{2,\mathrm{t}}^{\mathrm{curl}_\Gamma}$ to $L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ with inverse $w \to -\nu \times w$.

Definition 0.1. The Maxwell problem is to find a pair of radiating solution $(E, H) \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \Omega)$ to the Maxwell equations

$$\begin{aligned} \mathrm{curl}\, E - ikH &= 0 \\ \mathrm{curl}\, H + ikE &= 0 \end{aligned}$$

in $\mathbb{R}^3 \setminus \Omega$ with the boundary condition

$$\nu \times E = f$$

where $f \in H^{-\frac{1}{2}}(\mathrm{div}, \Gamma)$. The data-to-pattern operator $G : H^{-\frac{1}{2}}(\mathrm{div}, \Gamma) \to L_{\mathrm{t}}^2(\mathbb{S}^2)$ is defined by

$$Gf = E^\infty$$

where E^∞ denotes the far field pattern of the radiating solution E of the Maxwell problem.

Definition 0.2. The far field operator $F : L_{\mathrm{t}}^2(\mathbb{S}^2) \to L_{\mathrm{t}}^2(\mathbb{S}^2)$ is defined by

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} E^\infty(\hat{x}, \theta) g(\theta)\, d\sigma(\theta), \quad \hat{x} \in \mathbb{S}^2. \tag{0.14}$$

Proposition 0.6.
1. $F - F^* = \frac{ik}{8\pi} F^* F$, where F^* denotes the L^2-adjoint of F.
2. The scattering operator $S = I + \frac{ik}{8\pi^2} F$ is unitary.
3. F is normal.

Proof. Let $g, h \in L_{\mathrm{t}}^2(\mathbb{S}^2)$ and define the Hergoltz wave functions $v^{\mathrm{i}}, w^{\mathrm{i}}$ with density g, h respectively:

$$\begin{aligned} v^{\mathrm{i}}(x) &= \int_{\mathbb{S}^2} g(\theta) e^{ikx\cdot\theta}\, d\sigma(\theta), \qquad x \in \mathbb{R}^3 \\ w^{\mathrm{i}}(x) &= \int_{\mathbb{S}^2} h(\theta) e^{ikx\cdot\theta}\, d\sigma(\theta), \qquad x \in \mathbb{R}^3 \end{aligned}$$

Let v, w be solutions of the scattering problem corresponding to incident fields $v^{\mathrm{i}}, w^{\mathrm{i}}$, with scattered fields $v^{\mathrm{s}} = v - v^{\mathrm{i}}, w^{\mathrm{s}} = w - w^{\mathrm{i}}$ and far field patterns v^∞, w^∞ respectively. Apply Green theorem in $\Omega_R = \{x \in \mathbb{R}^3 \setminus \overline{\Omega} : |x| < R\}$ with sufficiently big R, together with the boundary condition we have

$$0 = \int_{\Omega_R} (v \Delta \overline{w} - \overline{w} \Delta v)\, dV \tag{0.15}$$

$$= \int_{|x|=R} (\overline{w} \times \mathrm{curl}\, v - v \times \mathrm{curl}\, \overline{w}) \cdot \nu\, d\sigma. \tag{0.16}$$

Decomposing $v = v^{\mathrm{i}} + v^{\mathrm{s}}$ and $w = w^{\mathrm{i}} + w^{\mathrm{s}}$, we split (0.16) into the sum of the following four parts:

$$\int_{|x|=R} \left(\overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{i}}}\right) \cdot \nu \, d\sigma, \tag{0.17}$$

$$\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, d\sigma, \tag{0.18}$$

$$\int_{|x|=R} \left(\overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{i}}}\right) \cdot \nu \, d\sigma, \tag{0.19}$$

$$\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, d\sigma. \tag{0.20}$$

The integral (0.17) vanishes by applying Green theorem in $B_R = \{x : |x| < R\}$. To evaluate the integral (0.18), we note by the radiation condition

$$\overline{w^{\mathrm{s}}} \times \hat{x} - \frac{1}{ik} \operatorname{curl} \overline{w^{\mathrm{s}}} = \mathcal{O}\left(\frac{1}{r^2}\right) \tag{0.21}$$

$$v^{\mathrm{s}} \times \hat{x} + \frac{1}{ik} \operatorname{curl} v^{\mathrm{s}} = \mathcal{O}\left(\frac{1}{r^2}\right) \tag{0.22}$$

and relations between scattered fields and far field patterns

$$\overline{w^{\mathrm{s}}} = \frac{e^{-ikr}}{4\pi r} \left\{ \overline{w^{\infty}} + \mathcal{O}\left(\frac{1}{r}\right) \right\}$$

$$v^{\mathrm{s}} = \frac{e^{ikr}}{4\pi r} \left\{ v^{\infty} + \mathcal{O}\left(\frac{1}{r}\right) \right\}$$

one obtains

$$\begin{aligned}
&(\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \hat{x} \\
&= ik \left(\overline{w^{\mathrm{s}}} \times (\hat{x} \times v^{\mathrm{s}}) + v^{\mathrm{s}} \times (\hat{x} \times \overline{w^{\mathrm{s}}})\right) \cdot \hat{x} \\
&= 2ik \left(\overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} - (\overline{w^{\mathrm{s}}} \cdot \hat{x})(v^{\mathrm{s}} \cdot \hat{x})\right) \\
&= 2ik \, \overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} \\
&= \frac{ik}{8\pi^2 r^2} \overline{w^{\infty}} \cdot v^{\infty} + \mathcal{O}\left(\frac{1}{r^3}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, d\sigma \\
&\longrightarrow \frac{ik}{8\pi^2} \int_{\mathbb{S}^2} \overline{w^{\infty}} \cdot v^{\infty} \, d\sigma = \frac{ik}{8\pi^2} (Fg, Fh)_{L^2(\mathbb{S}^2)}
\end{aligned}$$

To evaluate the integral (0.19), one note that it can be rearranged as

$$\int_{|x|=R} \left(\overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{i}}}\right) \cdot \nu \, d\sigma \tag{0.23}$$

$$= - \int_{|x|=R} (\hat{x} \times \operatorname{curl} v^{\mathrm{s}}) \cdot \overline{w^{\mathrm{i}}} + (\hat{x} \times v^{\mathrm{s}}) \cdot \operatorname{curl} \overline{w^{\mathrm{i}}} \, d\sigma \tag{0.24}$$

Substitute

$$\overline{w^{\mathrm{i}}}(x) = \int_{\mathbb{S}^2} h(\theta) e^{-ikx\cdot\theta}\, d\sigma(\theta),$$
$$\operatorname{curl} \overline{w^{\mathrm{i}}}(x) = ik \int_{\mathbb{S}^2} (h(\theta)\times\theta)\, e^{-ikx\cdot\theta}\, d\sigma(\theta)$$

into (0.24), the integral becomes

$$-\int_{|x|=R} (\hat{x}\times \operatorname{curl} v^{\mathrm{s}}) \cdot \int_{\mathbb{S}^2} h(\theta) e^{-ikx\cdot\theta}\, d\sigma(\theta)\, d\sigma(x)$$
$$-\int_{|x|=R} (\hat{x}\times v^{\mathrm{s}})\cdot ik \int_{\mathbb{S}^2} (h(\theta)\times\theta)\, e^{-ikx\cdot\theta}\, d\sigma(\theta)\, d\sigma(x). \quad (0.25)$$

From $h(\theta)\cdot\theta = 0$ and $\theta\cdot\theta = 1$, by formulae

$$a\times(b\times c) = b\,(a\cdot c) - c\,(a\cdot b)$$
$$a\cdot(b\times c) = -b\cdot(a\times c)$$

we have

$$\begin{aligned} h(\theta)\cdot(\hat{x}\times \operatorname{curl} v^{\mathrm{s}}) &= h(\theta)\cdot\{(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) - \theta\,(\theta\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}))\} \\ &= h(\theta)\cdot\{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta)\} \end{aligned}$$

and

$$(\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta) = h(\theta)\cdot(\theta\times(\hat{x}\times v^{\mathrm{s}}))$$

Substitute into (0.25), the value of the integral (0.19) is

$$\begin{aligned} &-\int_{\mathbb{S}^2}\int_{|x|=R} \{h(\theta)\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) + ik\,(\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta)\}\, e^{-ikx\cdot\theta}\, d\sigma(x)\, d\sigma(\theta) \\ &= -\int_{\mathbb{S}^2} h(\theta)\cdot \int_{|x|=R} \{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta) + ik\,\theta\times(\hat{x}\times v^{\mathrm{s}})\}\, e^{-ikx\cdot\theta}\, d\sigma(x)\, d\sigma(\theta) \\ &\longrightarrow -\left(Fg, h\right)_{L^2(\mathbb{S}^2)}. \end{aligned}$$

By the same token, the integral (0.20) is $(g, Fh)_{L^2(\mathbb{S}^2)}$. Hence

$$0 = (g, Fh)_{L^2(\mathbb{S}^2)} - (Fg, h)_{L^2(\mathbb{S}^2)} + \frac{ik}{8\pi^2}\,(Fg, Fh)_{L^2(\mathbb{S}^2)},$$

the identity

$$F - F^* = \frac{ik}{8\pi^2} F^* F$$

follows.

To see that S is unitary, we compute

$$\begin{aligned} S^*S &= \left(I - \frac{ik}{8\pi^2}F^*\right)\left(I + \frac{ik}{8\pi^2}F\right) \\ &= I + \frac{ik}{8\pi^2}F - \frac{ik}{8\pi^2}F^* + \frac{k^2}{64\pi^2}F^*F \\ &= I. \end{aligned}$$

Thus S is injective as well as surjective, for S is a compact perturbation of the identity. Therefore $S^* = S^{-1}$ and $SS^* = I$. Comparing S^*S and SS^* we can see that $F^*F = FF^*$, hence F is normal.

$\square$

Proposition 0.7.

$$F = -GN^*G^*.$$

Proof. Define auxiliary operator $\mathcal{H} : L^2_t(\mathbb{S}^2) \to H^{-\frac{1}{2}}(\mathrm{div}, \Gamma)$ as

$$(\mathcal{H}g)(x) = \nu(x) \times \int_{\mathbb{S}^2} g(\theta) e^{ikx\cdot\theta}\, d\sigma(\theta), \quad x \in \Gamma. \tag{0.26}$$

The adjoint operator $\mathcal{H}^* : H^{-\frac{1}{2}}(\mathrm{curl}, \Gamma) \to L^2_t(\mathbb{S}^2)$ is

$$(\mathcal{H}^* f)(\theta) = \theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right), \quad \theta \in \mathbb{S}^2. \tag{0.27}$$

This can be verified by

$$\begin{aligned}
\langle f, \mathcal{H}g\rangle &= \int_\Gamma f(x) \cdot \overline{\left\{\nu(x) \times \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, d\sigma(\theta)\right\}}\, d\sigma(x) \\
&= \int_\Gamma \int_{\mathbb{S}^2} f(x) \cdot (\nu(x) \times \overline{g(\theta)})\, e^{-ikx\cdot\theta}\, d\sigma(\theta)\, d\sigma(x) \\
&= \int_{\mathbb{S}^2} \int_\Gamma f(x) \cdot (\nu(x) \times \overline{g(\theta)})\, e^{-ikx\cdot\theta}\, d\sigma(x)\, d\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (f(x) \times \nu(x)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\, d\sigma(x)\, d\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (\theta \times ((f(x) \times \nu(x)) \times \theta)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta} d\sigma(x)\, d\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\theta \times \left(\int_\Gamma (f(x) \times \nu(x))\, e^{-ikx\cdot\theta}\, d\sigma(x) \times \theta\right)\right\} \cdot \overline{g(\theta)}\, d\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\left(\theta \times \int_\Gamma (f(x) \times \nu(x))\, e^{-ikx\cdot\theta}\, d\sigma(x)\right) \times \theta\right\} \cdot \overline{g(\theta)}\, d\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, d\sigma(x)\right)\right\} \cdot \overline{g(\theta)}\, d\sigma(\theta) \\
&= \langle \mathcal{H}^* f, g\rangle.
\end{aligned}$$

Given tangential $f(x)$, define $u(x)$ by

$$u(x) = \mathrm{curl}\,\mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi(x, y)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

From the asymptotic relation

$$\mathrm{curl}\,\mathrm{curl}_x \left\{a(y) \frac{e^{ik|x-y|}}{|x-y|}\right\} = k^2 \frac{e^{ik|x|}}{|x|} \left\{e^{-ik\hat{x}\cdot y}\, \hat{x} \times (\hat{x} \times a(y)) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}$$

the far field pattern of u can be seen as $\mathcal{H}^* f$.

Define the electric dipole operator N as

$$(Nf)(x) = \nu(x) \times \operatorname{curl}\operatorname{curl} \int_\Gamma (\nu(y) \times f(y))\, \Phi(x,y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{0.28}$$

Then

$$\mathcal{H}^* f = GNf. \tag{0.29}$$

We have

$$F = -G\mathcal{H}. \tag{0.30}$$

hence $F = -G\mathcal{H} = -GN^*G^*$. □

Proposition 0.8. $\Im\langle N\varphi, \varphi\rangle \geqslant 0$.

Proof. Define

$$v(x) = \operatorname{curl} \int_\Gamma \nu(y) \times \varphi(y)\, \Phi(x,y)\, \mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3 \setminus \Gamma. \tag{0.31}$$

Note that

$$\begin{aligned} v_\pm(x) &= \mathrm{pv} \int_\Gamma \nabla_x \Phi(x,y) \times (\nu(y) \times \varphi(y))\ \mathrm{d}\sigma(y) \mp \frac{1}{2}\nu(x) \times (\nu(x) \times \varphi(x)) \\ &= \mathrm{pv} \int_\Gamma \nabla_x \Phi(x,y) \times (\nu(y) \times \varphi(y))\ \mathrm{d}\sigma(y) \pm \frac{1}{2}\varphi(x) \end{aligned}$$

and $\operatorname{div} v = 0, \Delta v + k^2 v = 0$.

set $a = \overline{v}, b = v$ in vector Green formula

$$\int_\Omega a \cdot \Delta b + \operatorname{curl} a \cdot \operatorname{curl} b + \operatorname{div} a \operatorname{div} b = \int_\Gamma -(\nu \times \operatorname{curl} b) \cdot a + (\nu \cdot a) \operatorname{div} b$$

we can see that

$$\begin{aligned} \langle N\varphi, \varphi\rangle &= \langle \nu \times \operatorname{curl} v, v_+ - v_- \rangle \\ &= \int_\Gamma \nu \times \operatorname{curl} v \cdot (\overline{v_+} - \overline{v_-})\, \mathrm{d}\sigma \\ &= \int_\Gamma \nu \times \operatorname{curl} v \cdot \overline{v_+}\, \mathrm{d}\sigma - \int_\Gamma \nu \times \operatorname{curl} v \cdot \overline{v_-}\, \mathrm{d}\sigma \\ &= -\int_{\Omega \cup B_R} k^2 |v|^2 - |\operatorname{curl} v|^2\, dV + \int_{|x|=R} \hat{x} \times \operatorname{curl} v \cdot \overline{v}\, d\sigma \\ &= -\int_{\Omega \cup B_R} k^2 |v|^2 - |\operatorname{curl} v|^2\, dV + ik \int_{|x|=R} |v|^2\, d\sigma + \mathcal{O}\left(\frac{1}{R}\right) \end{aligned}$$

Take the imaginary part and let $R \to \infty$, we have

$$\Im\langle N\varphi, \varphi\rangle = k \lim_{R\to\infty} \int_{|x|=R} |v|^2\, d\sigma = \frac{k}{16\pi^2} \int_{\mathbb{S}^2} |v^\infty|^2\, d\sigma \geqslant 0.$$

□

Proposition 0.9. Given a bounded Lipschitz domain Ω, the followings hold:

1. There exists a regular family of cones $\{\zeta\}$.

2. There exists a sequence of C^∞ domains $\Omega_i \subset \Omega$ and corresponding homeomorphisms $\Lambda_j : \Gamma \to \Gamma_i$ such that $\sup_{x\in\Gamma} |\Lambda_j(x) - x| \to 0$ as $j \to \infty$ and for all j and all $x \in \Gamma$, $\Lambda_j(x) \in \zeta(x)$.

3. There exist positive functions $\omega_j : \Gamma \to \mathbb{R}^+$ bounded away from zero and infinity uniformly in j such that

   (a) For any measurable set $V \subset \Gamma$
   $$\int_V \omega_j \, d\sigma = \int_{\Lambda_j(V)} d\sigma_j.$$

   (b) $\omega_j(x) \to 1$ pointwise a.e. for $x \in \Gamma$.

4. $\nu(\Lambda_j(x)) \to \nu(x)$ pointwise a.e. for $x \in \Gamma$.

5. There exists a real-valued C^∞ vector field h such that for all j and $x \in \Gamma$, $\nu(\Lambda_j(x)) \cdot h(\Lambda_j(x)) \geqslant \kappa > 0$, where κ depends on the Lipschitz character of Ω. Without loss of generality, $\kappa < 1$.

Lemma 0.1 (Rellich identity)**.** For a complex-valued $C^\infty(\overline{\Omega})$ vector field E and a real-valued $C^\infty(\mathbb{R}^3)$ vector field h

$$\int_\Gamma \Big\{ \frac{1}{2}|E|^2 (h\cdot\nu) - \Re\big((\overline{E}\cdot h)(E\cdot\nu)\big)\Big\} \,\mathrm{d}\sigma \\ = \int_\Omega \Re\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\,\mathrm{d}V, \quad (0.32)$$

where $\overline{E}\cdot(\nabla h)E$ denotes the quadratic form $\Sigma_{i,j}(D_i h_j)E_i\overline{E_j}$.

Proof. It is evident from

$$\operatorname{div}\Big\{\frac{1}{2}|E|^2 h - \Re\big((\overline{E}\cdot h)E\big)\Big\} \\ = \Re\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}$$

and Divergence theorem. □

Lemma 0.2. For a complex-valued $C^\infty(\overline{\Omega})$ vector field E

$$\int_\Gamma |E|^2 \,\mathrm{d}\sigma \lesssim \int_\Gamma |E_\mathrm{n}|^2 \,\mathrm{d}\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 \,\mathrm{d}V \quad (0.33)$$

$$\int_\Gamma |E|^2 \,\mathrm{d}\sigma \lesssim \int_\Gamma |E_\mathrm{t}|^2 \,\mathrm{d}\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 \,\mathrm{d}V. \quad (0.34)$$

If $E \in C^\infty(\overline{\Omega_+})$ and decays at infinity then the above hold with Ω replaced by Ω_+.

Proof. Let h be the real-valued vector field which satisfies proposition 0.9, item (5), i.e. $h \cdot \nu \geqslant \kappa > 0$ on Γ. Decomposing E, h into mutually orthogonal parts $E = E_{\mathrm{t}} + E_{\mathrm{n}}$, $h = h_{\mathrm{t}} + h_{\mathrm{n}}$, we have

$$\begin{aligned}&\frac{1}{2}|E|^2(h\cdot\nu) - \Re\big((\overline{E}\cdot h)(E\cdot\nu)\big)\\ &\qquad\qquad = \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu) - \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu) - \Re\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big),\end{aligned}$$

thus the Rellich identity (0.32) is rewritten as

$$\int_\Gamma \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu)\,\mathrm{d}\sigma = \int_\Gamma \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu)\,\mathrm{d}\sigma + \Theta_1 + \Theta_2, \tag{0.35}$$

where

$$\begin{aligned}\Theta_1 &:= \int_\Gamma \Re\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big)\,\mathrm{d}\sigma,\\ \Theta_2 &:= \int_\Omega \Re\Big\{\frac{1}{2}|E|^2\operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\,\mathrm{d}V\end{aligned}$$

In view of (0.35) and $h \cdot \nu \geqslant \kappa > 0$ we have

$$\frac{1}{2}\kappa\int_\Gamma |E_{\mathrm{t}}|^2\,\mathrm{d}\sigma \leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{n}}|^2\,\mathrm{d}\sigma + \Theta_1 + \Theta_2. \tag{0.36}$$

By Young's inequality

$$ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \quad \forall\varepsilon > 0$$

(0.36) becomes

$$\int_\Gamma |E|^2\,\mathrm{d}\sigma \lesssim \int_\Gamma |E_{\mathrm{n}}|^2\,\mathrm{d}\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,\mathrm{d}V \tag{0.37}$$

Similarly, from (0.35) and (5) $h \cdot \nu \geqslant \kappa > 0$ we have

$$\begin{aligned}\frac{1}{2}\kappa\int_\Gamma |E_{\mathrm{n}}|^2\,\mathrm{d}\sigma &\leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{t}}|^2\,\mathrm{d}\sigma - \Theta_1 - \Theta_2\\ &\leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{t}}|^2\,\mathrm{d}\sigma + |\Theta_1| + |\Theta_2|,\end{aligned} \tag{0.38}$$

hence by Young's inequality (0.38) becomes

$$\int_\Gamma |E|^2\,\mathrm{d}\sigma \lesssim \int_\Gamma |E_{\mathrm{t}}|^2\,\mathrm{d}\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,\mathrm{d}V. \tag{0.39}$$

Once by Young's inequality

$$\int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,\mathrm{d}V \lesssim \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\,\mathrm{d}V,$$

and we may rewrite (0.37), (0.39) into (0.33), (0.34) respectively. □

Lemma 0.3. For the complex-valued $C^\infty(\overline{\Omega})$ vector field E which satisfies $(\triangle + k^2)E = 0$ and $\operatorname{div} E = 0$ in Ω,

$$\|E\|_{L_2(\Gamma)} + \|\operatorname{curl} E\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} E\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)}$$

Proof. Setting $a = \overline{E}$ and $b = E$ in vector Green's theorem

$$\int_\Omega a \triangle b + \operatorname{curl} a \cdot \operatorname{curl} b + \operatorname{div} a \cdot \operatorname{div} b = \int_\Gamma (\nu \times a) \cdot \operatorname{curl} b + (\nu \cdot a) \operatorname{div} b$$

we have

$$\int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E + (\overline{E} \cdot \nu) \operatorname{div} E \,\mathrm{d}\sigma = \int_\Omega |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - k^2 |E|^2 \,\mathrm{d}V.$$

The above identity becomes

$$\int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 \,\mathrm{d}V$$
$$\lesssim \left| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \right| + \int_\Gamma |E \cdot \nu| |\operatorname{div} E| \,\mathrm{d}\sigma.$$

Once by $|E \cdot \nu| \leqslant |E|$ and Young's inequality

$$\int_\Gamma |E \cdot \nu| |\operatorname{div} E| \,\mathrm{d}\sigma \leqslant (\text{small}) \int_\Gamma |E|^2 \,\mathrm{d}\sigma + (\text{large}) \int_\Gamma |\operatorname{div} E|^2 \,\mathrm{d}\sigma,$$

which turns (0.33) into

$$\int_\Gamma |\nu \times E|^2 \,\mathrm{d}\sigma \lesssim \int_\Gamma |E \cdot \nu|^2 + |\operatorname{div} E|^2 \,\mathrm{d}\sigma + \left| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \right|. \tag{0.40}$$

Together with the result of lemma 0.2, we have

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{n}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_\mathrm{t}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}, \\ \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_\mathrm{t}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}. \end{aligned} \tag{0.41}$$

By writing $H = \frac{1}{ik} \operatorname{curl} E$, (0.41) becomes

$$\|E\|_{L_2(\Gamma)} \lesssim \|E_\mathrm{n}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)}, \tag{0.42}$$
$$\|E\|_{L_2(\Gamma)} \lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)}. \tag{0.43}$$

From $\operatorname{curl} \operatorname{curl} E = -\triangle E + \nabla \operatorname{div} E$ we are free to permute E and H in (0.42), (0.43) and obtain

$$\|H\|_{L_2(\Gamma)} \lesssim \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}, \tag{0.44}$$
$$\|H\|_{L_2(\Gamma)} \lesssim \|H_\mathrm{t}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}. \tag{0.45}$$

By (0.43) and (0.44),

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)} \\ &\lesssim \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}. \end{aligned} \tag{0.46}$$

From (0.46), (0.44) and $\|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)} \lesssim \|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)}$, we have

$$\|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \approx \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)}. \tag{0.47}$$

Once by permutting E and H in (0.47) we have

$$\|H\|_{L_2(\Gamma)} + \|E\|_{L_2(\Gamma)} \approx \|H_{\mathrm{t}}\|_{L_2(\Gamma)} + \|E_{\mathrm{n}}\|_{L_2(\Gamma)}, \tag{0.48}$$

By $\|\cdot\|_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)} \equiv \|\cdot\|_{L_2(\Gamma)} + \|\operatorname{div}_\Gamma(\cdot)\|_{L_2(\Gamma)}$ and $\operatorname{div}_\Gamma(\nu \times E) = -\nu \cdot \operatorname{curl} E$, (0.48) is written as

$$\|E\|_{L_2(\Gamma)} + \|\operatorname{curl} E\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} E\|_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)} \tag{0.49}$$

as claimed. □

Proposition 0.10. $-\langle N_i\varphi, \varphi\rangle \geqslant c\|\varphi\|^2$.

Proof.

$$-\langle N_i\varphi, \varphi\rangle = \int_{\Omega\cup B_R} |v|^2 + |\operatorname{curl} v|^2 \, dV + \int_{|x|=R} |v|^2 \, d\sigma + \mathcal{O}\left(\frac{1}{R}\right)$$

As $R \to \infty$,

$$-\langle N_i\varphi, \varphi\rangle = \int_{\mathbb{R}^3} |v|^2 + |\operatorname{curl} v|^2 \, dV \geqslant \int_\Gamma |v|^2 + |\operatorname{curl} v|^2 \, d\sigma.$$

Recall that

$$v = \operatorname{curl} \int_\Gamma \nu(y) \times \varphi(y)\, \Phi(x,y)\, \mathrm{d}\sigma(y)$$

Set $E = v$ in lemma 0.3, we have

$$\|v\|_{L_2(\Gamma)} + \|\operatorname{curl} v\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} v\|_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)}$$

Hence

$$-\langle N_i\varphi, \varphi\rangle \geqslant c\,\|\nu \times \operatorname{curl} v\|^2_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)} = c\,\|N_i\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)} \geqslant c\,\|\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}.$$

□

Proposition 0.11. For $z \in \mathbb{R}^3$ and a fixed $d \in \mathbb{S}^2$, define $\varphi_z \in L^2(\mathbb{S}^2)$ by

$$\varphi_z(\hat{x}) = ik\,(\hat{x} \times d)e^{ik\hat{x}\cdot z} \qquad \hat{x} \in \mathbb{S}^2,$$

then φ_z belongs to the range of G if and only if $z \in \Omega$.

Proof. Assume first $z \in \Omega$. For $x \in \mathbb{R}^3 \setminus \Omega$ define

$$v(x) = \operatorname{curl}_x d\,\Phi(x,z) = \operatorname{curl}_x d\,\frac{e^{ik|x-z|}}{4\pi|x-z|}$$

and $f = v|_\Gamma$. The far field pattern of v, denoted by v^∞, is

$$v^\infty(\hat{x}) = ik\,(\hat{x} \times d)\,e^{ik\hat{x}\cdot z}, \qquad \hat{x} \in \mathbb{S}^2,$$

which is identical to φ_z. From $Gf = v^\infty = \varphi_z$, φ_z belongs to the range of G.

Now assume $z \notin \Omega$ and there exists f with $Gf = \varphi_z$. Let v be the radiating solution of the Maxwell problem with boundary data f and $v^\infty = Gf$ be the far field pattern of v. Note that the far field pattern of $\operatorname{curl} d\,\Phi(\cdot, z)$ is φ_z, from Rellich lemma $v(x) = \operatorname{curl} d\,\Phi(x, z)$ for all x outside of any sphere which contains both z and Ω. By analytic continuation, v and $\operatorname{curl} d\,\Phi(\cdot, z)$ coincide on $\mathbb{R}^3 \setminus (\overline{\Omega} \cup \{z\})$. But if $z \notin \overline{\Omega}$, then $\operatorname{curl} d\,\Phi(x, z)$ is singular on $x = z$, while v is analytic on $\mathbb{R}^3 \setminus \overline{\Omega}$, a contradiction. Otherwise if $z \in \Gamma$, then $x \mapsto \operatorname{curl} d\,\Phi(x, z)$ for $x \in \Gamma, x \neq z$, is in $H^{\frac{1}{2}}(\Gamma)$. But $\operatorname{curl} d\,\Phi(x, z)$ does not belong to $H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \Omega)$ or $H(\operatorname{curl}, \Omega)$, for $\operatorname{curl} \Phi(x, z) = \mathcal{O}\left(1/|x - z|^2\right)$ if $x \to z$.

□

0.5 An Illustration Using Spherical Wave Expansion

In this section we follow the notations and treatments in [?] closely.

The spherical Bessel and Hankel functions which denoted by $j_l(x)$, $n_l(x)$, $h_l(x)$, $h_l^{(1)}(x)$ are defined as

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \tag{0.50}$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x) \tag{0.51}$$

$$h_l(x) = \sqrt{\frac{\pi}{2x}} \left(J_{l+\frac{1}{2}}(x) + iN_{l+\frac{1}{2}}(x) \right) \tag{0.52}$$

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} \left(J_{l+\frac{1}{2}}(x) - iN_{l+\frac{1}{2}}(x) \right) \tag{0.53}$$

The spherical Bessel functions satisfy the recursion formulae

$$f_l(x) = \frac{x}{2l+1} \left(f_{l-1}(x) + f_{l+1}(x) \right) \tag{0.54}$$

$$f_l'(x) = \frac{1}{2l+1} \left(l f_{l-1}(x) - (l+1) f_{l+1}(x) \right) \tag{0.55}$$

where $f_l(x)$ is any one of the function $j_l(x)$, $n_l(x)$, $h_l(x)$, $h_l^{(1)}(x)$.

The orbital angular momentum operator $\mathbf{L}$ is defined by

$$\mathbf{L} = \frac{1}{i}\, x \times \nabla \tag{0.56}$$

where x is the position vector.

Define the operators L_x, L_y, L_z to be the cartesian components of the orbital angular-momentum operator $\mathbf{L}$ respectively, and let $L^2 = L_x^2 + L_y^2 + L_z^2$.

$$-\left\{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_l^m=l(l+1)Y_l^m \tag{0.57}$$

$$L_+=L_x+iL_y=e^{i\varphi}\left(\frac{\partial}{\partial\vartheta}+i\cot\vartheta\frac{\partial}{\partial\varphi}\right) \tag{0.58}$$

$$L_-=L_x-iL_y=e^{-i\varphi}\left(-\frac{\partial}{\partial\vartheta}+i\cot\vartheta\frac{\partial}{\partial\varphi}\right) \tag{0.59}$$

$$L_z=-i\frac{\partial}{\partial\varphi} \tag{0.60}$$

The vector spherical harmonic $X_l^m(\vartheta,\varphi)$ is defined by

$$X_l^m(\vartheta,\varphi)=\frac{1}{\sqrt{l(l+1)}}\mathbf{L}\,Y_l^m(\vartheta,\varphi) \tag{0.61}$$

With $\hat{x}=\frac{x}{\|x\|}$, we have the orthogonal relations

$$\int\overline{X_l^m}\cdot X_{l'}^{m'}\,d\Omega=\delta_{ll'}\delta_{mm'} \tag{0.62}$$

$$\int\overline{X_l^m}\cdot(\hat{x}\times X_{l'}^{m'})\,d\Omega=0 \tag{0.63}$$

$$\hat{x}\cdot X_l^m(\vartheta,\varphi)=0, \tag{0.64}$$

$$L_+Y_l^m=\sqrt{(l-m)(l+m+1)}Y_l^{m+1} \tag{0.65}$$

$$L_-Y_l^m=\sqrt{(l+m)(l-m+1)}Y_l^{m-1} \tag{0.66}$$

$$L_zY_l^m=mY_l^m \tag{0.67}$$

$$\begin{aligned}&\nabla\times f_l(r)\,X_l^m(\vartheta,\varphi)\\&\quad=i\hat{x}\sqrt{l(l+1)}\,\frac{f_l(r)}{r}\,Y_l^m(\vartheta,\varphi)+\frac{1}{r}\frac{\partial}{\partial r}(rf_l(r))\,\hat{x}\times X_l^m(\vartheta,\varphi)\end{aligned} \tag{0.68}$$

where $f_l(x)$ is any one of the function $j_l(x)$, $n_l(x)$, $h_l(x)$, $h_l^{(1)}(x)$.

$$g_l(kr)=A_l^{(1)}h_l^{(1)}(kr)+A_l^{(2)}h_l^{(2)}(kr) \tag{0.69}$$

$$\int\overline{f_l(r)X_l^m}\cdot g_l(r)X_{l'}^{m'}\,d\Omega=\overline{f_l}g_l\delta_{ll'}\delta_{mm'} \tag{0.70}$$

$$\int\overline{f_l(r)X_l^m}\cdot(\nabla\times g_l(r)X_{l'}^{m'})\,d\Omega=0 \tag{0.71}$$

$$\int \overline{\nabla \times f_l(r) X_l^m} \cdot (\nabla \times g_l(r) X_{l'}^{m'})\, d\Omega$$
$$= k^2 \delta_{ll'} \delta_{mm'} \left(\overline{f_l} g_l + \frac{1}{k^2 r^2} \frac{\partial}{\partial r} (r \overline{f_l} \frac{\partial}{\partial r}(r g_l)) \right), \quad (0.72)$$

where f_l, g_l are any of the spherical bessel functions.

The addition theorem for spherical harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \overline{Y_l^m(\vartheta', \varphi')} Y_l^m(\vartheta, \varphi) \quad (0.73)$$

where $\cos\gamma = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi')$

The multipole expansion of the plane wave is

$$E_{\mathrm{w}}(x) = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left(j_l(kr) X_l^{\pm 1} \pm \frac{1}{k} \nabla \times j_l(kr) X_l^{\pm 1} \right) \quad (0.74)$$

This is shown as follows. First note the Jacobi-Anger expansion

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\gamma) \quad (0.75)$$
$$= \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(kr) Y_l^0(\cos\gamma) \quad (0.76)$$

where γ is the angle between $\mathbf{k}$ and $\mathbf{x}$.

We consider an equivalent expansion for a circularly polarized plane wave with helicity $\pm$ along the z axis:

$$E(x) = (\varepsilon_1 \pm i\varepsilon_2) e^{ikz} \quad (0.77)$$
$$B(x) = \varepsilon_3 \times E = \mp i E \quad (0.78)$$

$$E(x) = \sum_{l,m} \left\{ a_\pm(l,m) j_l(kr) X_l^m + \frac{i}{k} b_\pm(l,m) \nabla \times j_l(kr) X_l^m \right\} \quad (0.79)$$
$$B(x) = \sum_{l,m} \left\{ \frac{-i}{k} a_\pm(l,m) j_l(kr) X_l^m + b_\pm(l,m) \nabla \times j_l(kr) X_l^m \right\} \quad (0.80)$$

From the orthogonality of X_l^m, we have

$$a_\pm(l,m) j_l(kr) = \int \overline{X_l^m} \cdot E(x)\, d\Omega \quad (0.81)$$
$$b_\pm(l,m) j_l(kr) = \int \overline{X_l^m} \cdot B(x)\, d\Omega \quad (0.82)$$

In view of the expression of E, B and the definition of X_l^m, after some manipulation we observed

$$a_\pm(l,m)j_l(kr) = \frac{1}{\sqrt{l(l+1)}}\int \overline{L_\mp Y_l^m} e^{ikz}\, d\Omega \tag{0.83}$$

$$a_\pm(l,m)j_l(kr) = \frac{\sqrt{(l\pm m)(l\mp m+1)}}{\sqrt{l(l+1)}}\int \overline{Y_l^{m\pm 1}} e^{ikz}\, d\Omega \tag{0.84}$$

Insert the Jacobi-Anger expansion for e^{ikz}, the orthogonality of Y_l^m leads to

$$a_\pm(l,m) = i^l\sqrt{4\pi(2l+1)}\delta_{m,\pm 1} \tag{0.85}$$

From $B = \mp iE$, we obtain immediately

$$b_\pm(l,m) = \mp i a_\pm(l,m) \tag{0.86}$$

The scattered electric field is

$$E_{\mathrm{s}}(x) = \frac{1}{2}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\cdot \left(\frac{j_l(k)}{h_l(k)}\, h_l(kr)X_l^{\pm 1} \pm \frac{1}{k}\frac{kj_l'(k)+j_l(k)}{kh_l'(k)+h_l(k)}\,\nabla\times h_l(kr)\, X_l^{\pm 1}\right) \tag{0.87}$$

The far field pattern of the scattered electric field is

$$E_\infty(\hat{x}) = \frac{-i}{2k}\sum_{l=1}^{\infty}\sqrt{4\pi(2l+1)}\cdot \left(\frac{j_l(k)}{h_l(k)}\,\hat{x}\times X_l^{\pm 1} \mp \frac{kj_l'(k)+j_l(k)}{kh_l'(k)+h_l(k)}\, X_l^{\pm 1}\right) \tag{0.88}$$

Hence $\{X_l^{\pm 1}, \hat{x}\times X_l^{\pm 1}\}$ are the eigenfunctions of the far field operator with corresponding eigenvalues $\{\frac{\pm i\sqrt{\pi(2l+1)}}{k}\frac{kj_l'(k)+j_l(k)}{kh_l'(k)+h_l(k)}, \frac{-i\sqrt{\pi(2l+1)}}{k}\frac{j_l(k)}{h_l(k)}\}$.

We wish to compute

$$\sum_m \frac{|\langle \hat{x}\times E_w, \phi_m\rangle|^2}{|\lambda_m|} \tag{0.89}$$

where the index m runs through the eigenpairs $\{\phi_m, \lambda_m\}$ of the far field operator and $\langle\cdot,\cdot\rangle$ denote the $L^2(\mathbb{S}^2)$ inner product. Note that

$$\hat{x}\times E_{\mathrm{w}}(x) = \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left(j_l(kr)\hat{x}\times X_l^{\pm 1} \mp \frac{1}{kr}\frac{\partial}{\partial r}(rj_l(kr))X_l^{\pm 1}\right) \tag{0.90}$$

In view of the vector formula

$$(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$$

we have

$$(\hat{x} \times X_l^{\pm 1}) \cdot (\hat{x} \times \overline{X_{l'}^{\pm 1}}) = (\hat{x} \cdot \hat{x})(X_l^{\pm 1} \cdot \overline{X_{l'}^{\pm 1}}) - (\hat{x} \cdot \overline{X_{l'}^{\pm 1}})(X_l^{\pm 1} \cdot \hat{x}) \tag{0.91}$$

$$= X_l^{\pm 1} \cdot \overline{X_{l'}^{\pm 1}} \tag{0.92}$$

Together with orthogonal relations (0.62) and (0.64), the infinite sum (0.89) becomes

$$\frac{4\sqrt{\pi}}{k} \sum_l \sqrt{2l+1} \left(\frac{|j_l(kr)|^2}{\left|\frac{j_l(k)}{h_l(k)}\right|} + \frac{|\frac{1}{kr}\frac{\partial}{\partial r}(r j_l(kr))|^2}{\left|\frac{k j_l'(k) + j_l(k)}{k h_l'(k) + h_l(k)}\right|} \right) \tag{0.93}$$

We wish to investigate the convergence of this sum.

Using the asymptotic relations of $j_l(k), h_l(k)$

$$j_l(k) = \frac{k^l}{1 \cdot 3 \cdots (2l+1)} \left(1 + \mathcal{O}\left(\frac{1}{l}\right)\right) \tag{0.94}$$

$$h_l(k) = \frac{1 \cdot 3 \cdots (2l-1)}{i k^{l+1}} \left(1 + \mathcal{O}\left(\frac{1}{l}\right)\right) \tag{0.95}$$

we have

$$\frac{j_l(k)}{h_l(k)} = -i \frac{k^{2l+1}}{(2l-1)!!(2l+1)!!} \left(1 + \mathcal{O}\left(\frac{1}{l}\right)\right) \tag{0.96}$$

$$\frac{k j_l'(k) + j_l(k)}{k h_l'(k) + h_l(k)} =? \left(1 + \mathcal{O}\left(\frac{1}{l}\right)\right) \tag{0.97}$$